Modeling multi-modal mobility in a coupled morning-evening commute framework that considers deadheading and flexible pooling

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16. Abstract

We develop a general equilibrium model to capture the complex interactions between solo driving, rideshare and ride-hailing services such as Uber and Lyft that allows travelers to switch between different transportation modes and allows passengers from different Origin-Destination (OD) pairs to share a ride together in a coupled morning-evening commute framework. The model is formulated as a variational inequality (VI), and reformulated as an equivalent mixed complementarity problem (MiCP). Then we prove the existence of an equilibrium solution, and provide the conditions on the model parameters under which the equilibrium will be unique. Furthermore, we prove that the travelers' disutility of our coupled model will not be worse than that of a decoupled modeling approach. The computational results on the Sioux-Falls network show that our model captures the possible mode switches between morning and evening commutes, as well as the detour of rideshare drivers to pick up or drop off passengers. Furthermore, our numerical examples demonstrate that modeling morning and evening commutes separately tends to overestimate the number of drivers and total Vehicle Hours Traveled (VHT) in the network when accounting for the coupling interaction effects between morning and evening commutes.

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Disclosure

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Abstract

We develop a general equilibrium model to capture the complex interactions between solo driving, rideshare and ride-hailing services such as Uber and Lyft that allows travelers to switch between different transportation modes and allows passengers from different Origin-Destination (OD) pairs to share a ride together in a coupled morning-evening commute framework. The model is formulated as a variational inequality (VI), and reformulated as an equivalent mixed complementarity problem (MiCP). Then we prove the existence of an equilibrium solution, and provide the conditions on the model parameters under which the equilibrium will be unique. Furthermore, we prove that the travelers' disutility of our coupled model will not be worse than that of a decoupled modeling approach. The computational results on the Sioux-Falls network show that our model captures the possible mode switches between morning and evening commutes, as well as the detour of rideshare drivers to pick up or drop off passengers. Furthermore, our numerical examples demonstrate that modeling morning and evening commutes separately tends to overestimate the number of drivers and total Vehicle Hours Traveled (VHT) in the network when accounting for the coupling interaction effects between morning and evening commutes.

Executive Summary

The rapid rise of ride-hailing services (provided by Uber, Lyft, Didi, Grab and Ola) and rideshare services (enabled by SCOOP, WAZE, Zipcar, and Turo) are transforming urban passenger transportation. (Note that in this report, rideshare refers to the mode of carpooling enabled by companies that distinguishes from traditional carpooling where no payment for the ride is made. It also differs from ride-hailing in terms of price and inconvenience cost.) While these new mobility services enriched the user experience by providing more mobility options, they also raise challenges for transportation planners in terms of analyzing the morning and evening commuting trips: (1) how to quantify travelers' possible mode switches between the morning commute and the evening commute; (2) how to capture the interactions between the various modes of transportation in the morning and evening commutes.

In facing these challenges, we propose a general equilibrium model that considers the joint travel decisions and interactions between solo driving, rideshare and ride-hailing in a coupled morning-evening commute modeling framework with the features of (i) providing simultaneously the results of traffic flows and travelers' mode choices; (ii) quantifying the possible mode switches across various transportation services between the morning and evening commutes; (iii) allowing for passengers from different Origin-Destination (OD) pairs to share a ride together; (iv) capturing the deadheading of ride-hailing vehicles; and (v) modeling the coupling interaction effects between morning and evening commutes. Formulated as a variational inequality and reformulated as an equivalent mixed complementarity problem, the main constraints of the general equilibrium model include: rideshare capacity constraints, demand satisfaction, flow conservation equations, extended user equilibrium conditions, etc.

Then we analyze the mathematical properties of our proposed model. We show an equilibrium solution exists for our proposed model, and provide the condition under which the solution is globally unique. Furthermore, we show that, under the same condition, the equilibrium will be locally unique even when a commonly used assumption in the literature is violated. In order to provide more theoretical insights for transportation planners, we compare the equilibrium solution from the our proposed coupled model with a decoupled modeling approach. It is proved that the travelers' disutility produced by our coupled model will not be worse than that of the decoupled model.

Finally, our proposed model is validated using the Sioux-Falls network. The results show that the proposed coupled morning-evening traffic equilibrium model is capable of capturing the mode

switches between morning and evening, and the detour of rideshare drivers. Specifically, 7.0% of rideshare drivers in the morning switch to be solo drivers in the evening; 30.9% of the rideshare drivers choose to take a detour for picking up or dropping off passengers in the morning. Our numerical examples show that considering morning and evening commutes separately tends to overestimate the travel cost, number of drivers and total Vehicle Hours Traveled (VHT) in the network. For example, the proposed model produces 7.8% fewer drivers and 15.4% less VHT in the system compared with a decoupled method when the rideshare price is higher in the evening commute than that of the morning commute. This is due to the coupling interaction effects between morning and evening commutes, e.g., rideshare passengers in the morning commute may switch to ride-hailing passengers in the evening commute. When treating the morning and evening commutes separately, we cannot capture these interactions.

1 Introduction

App-based transportation services, such as ride-hailing services (also called e-hailing or ridesourcing services in the literature, e.g., Ban et al., 2019) provided by Uber, Lyft, DiDi, Grab and Ola or casual rideshare[1] enabled by SCOOP, WAZE, Zipcar, and Turo are growing rapidly. For example, Uber has hit its milestone in 2018 to serve over 10 billion trips within more than 700 cities of 80 countries (Uber, 2018). There are over 75 million riders and 3.9 million drivers in total, producing more than 14.1 billion dollars of annual net revenue (Iqbal, 2020). These emerging transportation services are transforming the travel behavior of individuals and urban mobility patterns, and provide significant challenges to transportation planners and policy makers on how to assess the impact of these services on transportation systems, and how to facilitate or regulate these services.

Due to heavy traffic, commuters suffer from long travel delays in both the morning and evening commutes in many urban areas. Transportation planners should consider both commuting trips in their analysis, since the travel choices in one commute affect those of the other, but in practice they are rarely jointly analyzed. With the emerging transportation services, one challenge for transportation planners is to quantify travelers' possible mode switches between the morning commute and the evening commute. The ride-hailing and rideshare services provide more travel mode choices for commuters in both morning and evening commutes. For example, a person can combine a rideshare service in the morning, but use a ride-hailing service for the evening return trip to reduce the pairing cost, and provide more flexibility in evening trips. This capturing of mode switches is especially important if the travel cost data is different in the morning and evening times. For example, a traveler with a high inconvenience cost for rideshare in the evening, which may be due to the need to pick up their children from after-school activities, will not use this mode in the evening. Thus, an alternative option for these travelers is to use rideshare in the morning and to take ride-hailing in the evening.

Another challenge for transportation planners is to capture the interactions between the various modes of transportation in the morning and evening commutes. With rideshare and ride-hailing services, the choice of travelers in the morning/evening may influence that in the evening/morning. For example, a traveler may decide to drive in the morning if (s)he knows that it is very expensive to take a ride in the evening, which may be caused by the decrease of the evening supply in the

[1] In this report, rideshare refers to the mode of carpooling enabled by companies that distinguishes from traditional carpooling where no payment for the ride is made. It also differs from ride-hailing in terms of price and inconvenience cost.

rideshare and ride-hailing markets.

In recent years, researchers have included rideshare services in the traditional traffic assignment problem (Sheffi, 1985; Patriksson, 2015). Xu et al. (2015a) and Xu et al. (2015b) first proposed the traffic equilibrium models with rideshare services. Considering an Origin-Destination (OD) based surge pricing strategy, Ma et al. (2020) modeled a rideshare user equilibrium with ride-matching constraints. Li et al. (2020) studied a path-based rideshare equilibrium model to simultaneously produce route choices, mode choices, and matching decisions. Instead of using a mixed complementary formulation, Wang et al. (2021) established a convex programming formulation for the rideshare user equilibrium problem. Noruzoliaee and Zou (2022) formulated a rideshare user equilibrium model in an autonomous vehicle context. Some papers also extended the traditional traffic assignment problem by considering ride-hailing services. Ban et al. (2019) modeled the ride-hailing services in a general equilibrium model. Li et al. (2021) proposed a network equilibrium model with optimal spatial pricing for ride-hailing services. To better understand vacant trips generated by ride-hailing services, Xu et al. (2021) put forward a network equilibrium model to capture both cruising and deadheading trips of ride-hailing vehicles. Chen and Di (2022) formulated a ride-hailing network equilibrium model considering pooling options for passengers. Di and Ban (2019) proposed a general traffic equilibrium modeling framework which includes both rideshare and ride-hailing services.

To the best of our knowledge, there is no research to provide a general equilibrium model to capture the complex interactions between solo driving, rideshare, and ride-hailing in a coupled morning-evening commute framework. There are several reasons for developing a coupled morning-evening traffic equilibrium model to assist transportation planners in their decision making, especially considering the emerging rideshare and ride-hailing services: First, even for the same transportation network, traffic equilibria in the morning and evening commutes are not symmetrical due to different road networks for the morning and evening trips. Asymmetrical cost structures for the morning and evening commutes could further enlarge this difference; Second, with the competition or cooperation between various transportation modes, travelers may choose one type of commute mode in the morning period, and switch to a different type in the evening period, especially when the cost structures differ between the morning and the evening.

Although there are some papers to extend the bottleneck model (Vickrey, 1969; Li et al., 2020) as a Morning-evening Commute Problem (Zhang et al., 2008; Daganzo, 2013; Gonzales and Daganzo, 2013), for these papers the reason for considering both morning and evening commutes together is markedly different. The motivation for the bottleneck model is that the schedule penalty functions in the morning and evening vary. Finally, rideshare and ride-hailing services have not been considered in the Morning-evening Commute Problem.

In order to close the identified research gap for the traffic assignment problem, we propose a general equilibrium model that considers the joint travel decisions and interactions between solo driving, rideshare and ride-hailing in a coupled morning-evening commute modeling framework. The main contributions of this report are listed as follows:

- We develop a general equilibrium model framework to capture both rideshare and ride-hailing services between morning and evening commutes with the features of (i) providing simultaneously the results of traffic flows and travelers' mode choices; (ii) quantifying the possible mode switches across various transportation services between the morning and evening commutes; (iii) allowing for passengers from different OD pairs to share a ride together; and (iv) capturing the coupling interaction effects between morning and evening commutes.

- The proposed model is formulated as a variational inequality (VI), and reformulated as an equivalent mixed complementarity problem (MiCP). Then we show an equilibrium solution exists for our proposed model, and provide the condition under which the solution is unique. Moreover, we prove that the travelers' disutility produced by our coupled model will not be worse than that of a decoupled modeling approach.

- Then our proposed model is validated using the Sioux-Falls network. The experimental analysis shows the effects on Vehicle Hours Traveled (VHT), number of travelers to use each mode, number of mode switches, and number of detours as a function of the cost parameters such as rideshare inconvenience costs.

The remainder of this report is organized as follows. In Section 2, we provide the coupled morning-evening traffic equilibrium model with rideshare and ride-hailing services. In Section 3 and Section 4, we analyze the mathematical properties of the proposed model. In Section 5, experimental results are given to illustrate our model. Section 6 concludes this report and points out some possible directions for future research.

2 Mathematical Model

2.1 Problem description

We propose an extended traffic equilibrium model of morning and evening commutes, taking into account the emergent travel trends of rideshare and ride-hailing that offer alternative modes of travel supplementing the traditional mode of commuting: solo driving. The goal of the model is to study the morning and evening commute trip flows in the network caused by traffic congestion and the travelers' choices of commute types to minimize their disutilities. More importantly, our approach combines morning travel from an origin to a destination and evening return from the same destination (which therefore is the origin of the evening trip) to the morning's origin; this round trip is composed of a morning trip taken on a path and an evening trip taken on a possibly different (reverse) path with possibly a different mode. The round-trip path flows and mode choices encompass travelers' commute behavior; the equilibrium will determine the travelers' path and mode selections by equilibrating the round-trip path flows, morning mode choices and evening mode choices with the travelers' disutilities based on an extension of Wardrop's user equilibrium principle.

As illustrated in Fig. 1, there are different types of commuters: (1) drivers, labelled as d, including both solo and rideshare drivers labelled as sd and rd, respectively; namely $\mathrm{d} = \{\mathrm{sd}, \mathrm{rd}\}$; (2) passengers, labelled as p, including both rideshare and ride-hailing passengers labelled as rp and hp, respectively; thus $\mathrm{p} = \{\mathrm{rp}, \mathrm{hp}\}$; we use the letter $\mathrm{t} \in \{\mathrm{sd}, \mathrm{rd}, \mathrm{rp}, \mathrm{hp}\}$ as the generic label for these 4 types of travelers (i.e., commuters). In the morning/evening commute, drivers can choose to provide rideshare services if it is convenient for them. But drivers will not provide ride-hailing services since they also have their own destinations. In this scenario, ride-hailing services are provided by another group of drivers who are not commuters. Part of the complication of the model is for rideshare drivers to pick up and drop off passengers, possibly involving some detours of the driver's more direct routes to and from work place. As a result, drivers can switch roles between solo driver and rideshare driver during morning/evening commute. For various reasons, passengers may switch from one type of commute mode in the morning to a different type in the evening: rideshare passengers and ride-hailing passengers may switch among these two types.

Define $\mathcal{N}_0$ as the set of all nodes in a network. A morning OD pair $k = (i_k; j_k)$ joining nodes i_k and j_k in $\mathcal{N}_0$ becomes the OD pair $\bar{k} = (j_k; i_k)$ in the evening. That is to say, the origin and destination of morning OD pair $k \in \mathcal{K}$ becomes the destination and origin of evening OD pair $\bar{k}$, respectively. Each traveler will choose a morning path $p^{\,\mathrm{am}} \in \mathcal{P}_k^{\mathrm{am}}$ to go from home to workplace

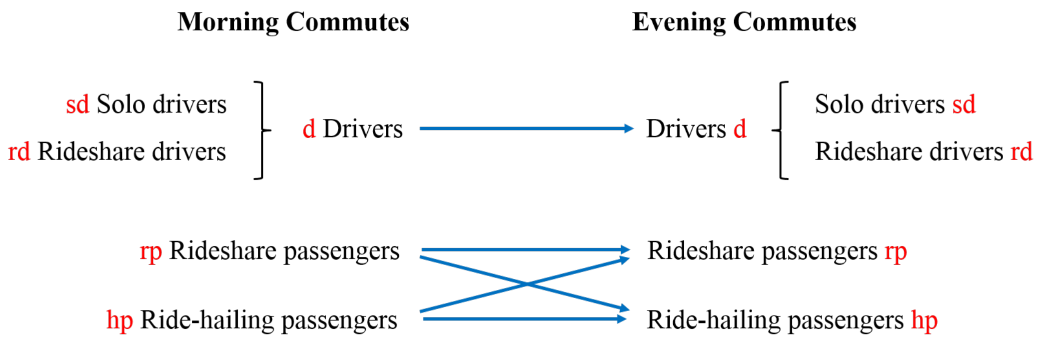

Figure 1. Mode choices and mode switches of morning and evening commutes.

and an evening path $p^{\text{pm}} \in \mathcal{P}_{\bar{k}}^{\text{pm}}$ to return home from workplace. Mathematically, if $\mathcal{P}_k$ denotes the set of all the paths throughout the entire day joining an OD pair k, then $\mathcal{P}_k = \mathcal{P}_k^{\text{am}} \times \mathcal{P}_{\bar{k}}^{\text{pm}}$. Let $\mathcal{P} = \{\mathcal{P}_k\}_{k \in \mathcal{K}}$. Based on a set of travel costs, under a set of assumptions and subject to traffic congestion, the model aims to determine a user equilibrium of trips for all the paths $p \in \mathcal{P}$ throughout the entire day. In this process, the model also determines the switch of the passenger types in the morning and the evening trips and also the switches among the drivers from solo to rideshare and vice versa.

2.2 Model notations and assumptions

Main notations used in this study are summarized as follows, including input sets and parameters in Table 1, and decision variables in Table 2 and Table 3.

In order to balance model realism and mathematical tractability, we assume that:

- The modeling context is static. That is to say, this is a model from a planner's perspective, instead of one at operational level.
- During the morning/evening commute, rideshare drivers may take a detour for picking up or dropping off rideshare passengers if needed. Thus, rideshare passengers with different OD pairs are allowed to share the same vehicle.
- A passenger does not change travel mode during his/her morning or evening trip. For example, during the morning commute, the passenger's entire trip must be either completely a rideshare trip or completely a ride-hailing trip.
- A driver in the morning commute will drive the car back home in the evening commute.
- Rideshare vehicles have the same passenger capacity, and each ride-hailing vehicle is assumed to pick up only one OD demand (passenger).

Table 1. Input sets and parameters

Network structure	
$\mathcal{G}_0 = (\mathcal{N}_0, \mathcal{A}_0)$	Original network with the node set $\mathcal{N}_0$ and the arc set $\mathcal{A}_0$, whose elements are denoted by $i, j \in \mathcal{N}_0$ and $a_0 \in \mathcal{A}_0$
$\mathcal{K} \subseteq \mathcal{N}_0 \times \mathcal{N}_0$	Set of OD pairs
$\bar{k}$	Evening return OD pair corresponding to morning OD pair $k \in \mathcal{K}$
$\mathcal{N}_0'$	Set of nodes of rideshare passengers
$\mathcal{N}_0''$	Set of nodes of ride-hailing passengers
$\mathcal{A}_\mathrm{t}$	Sets of arcs of traveler type t, $a_\mathrm{t} \in \mathcal{A}_\mathrm{t}$, $\mathrm{t} \in \{\mathrm{sd}, \mathrm{rd}, \mathrm{rp}, \mathrm{hp}\}$
$\mathcal{G} = (\mathcal{N}, \mathcal{A})$	Extended network with the node set $\mathcal{N}$ and the arc set $\mathcal{A}$; where $\mathcal{N} \triangleq \mathcal{N}_0 \cup \mathcal{N}_0' \cup \mathcal{N}_0''$ and $\mathcal{A} \triangleq \mathcal{A}_\mathrm{sd} \cup \mathcal{A}_\mathrm{rd} \cup \mathcal{A}_\mathrm{rp} \cup \mathcal{A}_\mathrm{hp}$
$\mathcal{T}_\mathrm{t}(a_0)$	Mapping an original arc $a_0 \in \mathcal{A}_0$ to its corresponding arc $a_\mathrm{t} \in \mathcal{A}_\mathrm{t}$, with $\mathrm{t} \in \{\mathrm{sd}, \mathrm{rd}, \mathrm{rp}, \mathrm{hp}\}$, i.e. $\mathcal{T}_\mathrm{t} : \mathcal{A}_0 \to \mathcal{A}_\mathrm{t}$
$\mathcal{T}_0(a_\mathrm{t})$	Mapping an arc $a_\mathrm{t} \in \mathcal{A}_\mathrm{t}$ with $\mathrm{t} \in \{\mathrm{sd}, \mathrm{rd}, \mathrm{rp}, \mathrm{hp}\}$ to the original arc $a_0 \in \mathcal{A}_0$ it is generated from, i.e. $\mathcal{T}_0 : \mathcal{A}_\mathrm{t} \to \mathcal{A}_0$
$\mathcal{P}_k^{\mathrm{am}}$	Set of paths for OD pair $k \in \mathcal{K}$ in the extended network in the morning
$\mathcal{P}_{\bar{k}}^{\mathrm{pm}}$	Set of paths for the return OD pair $\bar{k}$ in the extended network in the evening
$\mathcal{P}_k$	Set of paths for OD pair $k \in \mathcal{K}$ in the extended network throughout the entire day
$\mathcal{P}$	Set of all paths in the extended network throughout the entire day
Model parameters	
D_k	Total person-trip demand of OD pair $k \in \mathcal{K}$
M	Capacity in terms of number of rideshare passengers for each vehicle
$\delta_{a;p}$	Arc-path incidence indicator; $\delta_{a;p} = \begin{cases} 1 & \text{if path } p \text{ uses arc } a \\ 0 & \text{otherwise} \end{cases}$
$\mathrm{tt}_a(\bullet)$	The classic Bureau of Public Roads (BPR) travel time function for arc $a \in \mathcal{A}_0$ as a function of traffic flow on the arc: $\mathrm{tt}_a(\bullet) = t_a \left(1 + b \left[\frac{\bullet}{\alpha_a}\right]^4\right)$
t_a, α_a	Free flow travel time and flow capacity, respectively, of arc $a \in \mathcal{A}_0$, for calculation of travel time considering congestion
ψ	Conversion factor of time (minutes) to money (dollars)

Table 2. Primary and derived decision variables

f_a^{am}	Flow of travelers of arc $a \in \mathcal{A}$ in the morning
f_a^{pm}	Flow of travelers of arc $a \in \mathcal{A}$ in the evening
z_p	Flow of travelers of path $p \in \mathcal{P}$ through the entire day
$x_{a;\,k}^{\text{am}}$	Flow of travelers of arc $a \in \mathcal{A}$ and OD pair $k \in \mathcal{K}$ in the morning
$x_{a;\,\bar{k}}^{\text{pm}}$	Flow of travelers of arc $a \in \mathcal{A}$ and reverse OD pair $\bar{k}$ in the evening
u_k	Generalized (least) disutility of OD pair $k \in \mathcal{K}$
$\eta_a^{\pm;\text{am}}$	Morning shadow price of arc $a \in \mathcal{A}_0$, induced by morning rideshare capacity constraint
$\eta_a^{\pm;\text{pm}}$	Evening shadow price of arc $a \in \mathcal{A}_0$, induced by evening rideshare capacity constraint
$\mu_i^{k;\text{am}}$	Morning multiplier for OD pair $k \in \mathcal{K}$ and node $i \in \mathcal{N}$, induced by morning demand satisfaction constraint and morning flow conservation constraint
$\mu_i^{\bar{k};\text{pm}}$	Evening multiplier for reverse OD pair $\bar{k}$ and node $i \in \mathcal{N}$, induced by evening demand satisfaction constraint and evening flow conservation constraint
$\zeta_{a;\,k}$	Multiplier for OD pair $k \in \mathcal{K}$ and arc $a \in \mathcal{A}$, induced by driver flow conservation constraint

Table 3. Cost functions

$I_{a;\text{t}}^{\text{am}}(\boldsymbol{f}^{\text{am}})$	Morning inconvenience cost on arc $a \in \mathcal{A}_\text{t}$ experienced by commuter type $\text{t} \in \{\text{rd}, \text{rp}, \text{hp}\}$ as a function of morning arc flow $\boldsymbol{f}^{\text{am}} \triangleq \left\{ (f_a^{\text{am}})_{a \in \mathcal{A}_\text{t}} \right\}_{\text{t} \in \{\text{sd},\text{rd},\text{rp},\text{hp}\}}$
$I_{a;\text{t}}^{\text{pm}}(\boldsymbol{f}^{\text{pm}})$	Evening inconvenience cost on arc $a \in \mathcal{A}_\text{t}$ experienced by commuter type $\text{t} \in \{\text{rd}, \text{rp}, \text{hp}\}$ as a function of evening arc flow $\boldsymbol{f}^{\text{pm}} \triangleq \left\{ (f_a^{\text{pm}})_{a \in \mathcal{A}_\text{t}} \right\}_{\text{t} \in \{\text{sd},\text{rd},\text{rp},\text{hp}\}}$
$R_{a;\text{t}}^{\text{am}}(\boldsymbol{f}^{\text{am}})$	Morning rideshare payment/income or ride-hailing payment on arc $a_\text{t} \in \mathcal{A}_\text{t}$ experienced by commuter type $\text{t} \in \{\text{rd}, \text{rp}, \text{hp}\}$
$R_{a;\text{t}}^{\text{pm}}(\boldsymbol{f}^{\text{pm}})$	Evening rideshare payment/income or ride-hailing payment on arc $a_\text{t} \in \mathcal{A}_\text{t}$ experienced by commuter type $\text{t} \in \{\text{rd}, \text{rp}, \text{hp}\}$
$\text{tt}_a^{\text{am}}(\boldsymbol{f}^{\text{am}})$	Morning travel time on arc $a \in \mathcal{A}_0$
$\text{tt}_a^{\text{pm}}(\boldsymbol{f}^{\text{pm}})$	Evening travel time on arc $a \in \mathcal{A}_0$
$\text{tc}_a^{\text{am}}(\boldsymbol{f}^{\text{am}})$	Total cost on arc $a \in \mathcal{A}$ experienced by morning commuters
$\text{tc}_a^{\text{pm}}(\boldsymbol{f}^{\text{pm}})$	Total cost on arc $a \in \mathcal{A}$ experienced by evening commuters
$\text{TC}_p(\boldsymbol{z})$	Total cost on path $p \in \mathcal{P}$ experienced by commuters as a function of $\boldsymbol{z} \triangleq \{ z_p \}_{p \in \mathcal{P}}$

2.3 Constructing an extended network

To accommodate the switch between solo and rideshare drivers and the pick-up/dropoff locations, an extended network similar to the approach in Xu et al. (2015b) is constructed from the given network with splits of its node and arc sets. Specifically, each node of the original network is splitted to distinguish between drivers and passengers: a "driver arc" is splitted into a "solo-driver arc" and a "rideshare-driver arc"; and a "passenger arc", which is splitted into a "rideshare-passenger arc" and a "ride-hailing-passenger arc". The flows of solo-driver arcs, rideshare-driver arcs, and ride-hailing-passenger arcs will contribute to congestion. The totality of these splitted nodes and arcs defines the extended network.

Take Fig. 2 as an example, in Fig. 2(a), the original network consists of nodes i, j, ℓ, and arcs (i, j), (j, ℓ). Thus, the extended network in Fig. 2(b) consists of:

- $\mathcal{N}_0$: "driver" nodes i, j and ℓ;
- $\mathcal{N}_0'$: "rideshare-passenger" nodes i', j' and ℓ';
- $\mathcal{N}_0''$: "ride-hailing-passenger" nodes i'', j'' and ℓ'';
- $\mathcal{A}_{\text{sd}}$: "solo-driver" arcs (i, j) and (j, ℓ) (the one above in Fig. 2);
- $\mathcal{A}_{\text{rd}}$: "rideshare-driver" arcs (i, j) and (j, ℓ) (the one below in Fig. 2);
- $\mathcal{A}_{\text{rp}}$: "rideshare-passenger" arcs (i', j') and (j', ℓ');
- $\mathcal{A}_{\text{hp}}$: "ride-hailing-passenger" arcs (i'', j'') and (j'', ℓ'').

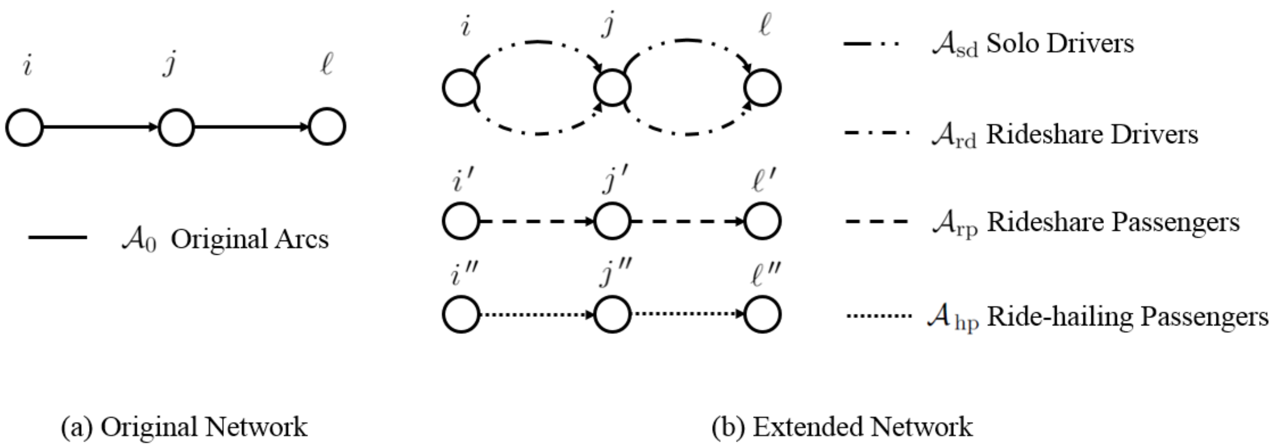

Figure 2. The original network and the extended network.

Note that these are not three disjoint networks since they are connected by the fixed demands, i.e., the sum of flows leaving the three splitted origin nodes (or entering the three splitted destination nodes) are fixed. As a result, travelers could choose to start at either node i if they want to drive, node i' if they wish to take the rideshare service, or node i'' if they prefer using the ride-hailing service. From the extended network in Fig. 2(b), we can notice that if travelers choose to start from driver node i, they may travel on arc $a_{\mathrm{sd}} \in \mathcal{A}_{\mathrm{sd}}$ and/or arc $a_{\mathrm{rd}} \in \mathcal{A}_{\mathrm{rd}}$ before reaching the destination node $\ell \in \mathcal{N}_0$, which is also a driver node. For example, in Fig. 2(b), a driver can be traveling along arc $(i, j) \in \mathcal{A}_{\mathrm{rd}}$ and then arc $(j, \ell) \in \mathcal{A}_{\mathrm{sd}}$. That is to say, (s)he drives with some passenger(s) to share a ride from node i to j, then drops off the passenger(s) at node j, and drives alone from j to ℓ. However, once departing from a driver node, travelers cannot switch to passenger arcs in $\mathcal{A}_{\mathrm{rp}}$ or $\mathcal{A}_{\mathrm{hp}}$. The reason is here we have the assumption that drivers will not leave their vehicles until they arrive at their destinations. Observing that there is no arcs connecting a rideshare-passenger node $\mathcal{N}_0'$ to a driver node $\mathcal{N}_0$ or a ride-hailing-passenger node $\mathcal{N}_0''$. Thus, if a traveler departs from a rideshare-passenger node i', (s)he is only allowed to travel on the rideshare passenger arcs $a_{\mathrm{rp}} \in \mathcal{A}_{\mathrm{rp}}$ before (s)he reaches the destination node $\ell' \in \mathcal{N}_0'$. This is because we assume that travelers cannot change their travel mode before reaching their destinations once they choose to be rideshare passengers. Similarly, once travelers depart from a ride-hailing passenger node i'', they are only allowed to travel on ride-hailing passenger arcs $a_{\mathrm{hp}} \in \mathcal{A}_{\mathrm{hp}}$ before they arrive at their destinations.

Since $\mathcal{A}_{\mathrm{sd}} \cup \mathcal{A}_{\mathrm{rd}} \subset \mathcal{N}_0 \times \mathcal{N}_0$, $\mathcal{A}_{\mathrm{rp}} \subset \mathcal{N}_0' \times \mathcal{N}_0'$, and $\mathcal{A}_{\mathrm{hp}} \subset \mathcal{N}_0'' \times \mathcal{N}_0''$, a path p will either only visit nodes in $\mathcal{N}_0$ using arcs in $\mathcal{A}_{\mathrm{sd}}$ or $\mathcal{A}_{\mathrm{rd}}$, or only visit nodes in $\mathcal{N}_0'$ using arcs in $\mathcal{A}_{\mathrm{rp}}$, or only visit nodes in $\mathcal{N}_0''$ using arcs in $\mathcal{A}_{\mathrm{hp}}$. Thus a path p can contain only arcs of $\mathcal{A}_{\mathrm{rp}}$, or only arcs of $\mathcal{A}_{\mathrm{hp}}$, or only arcs of $\mathcal{A}_{\mathrm{sd}} \cup \mathcal{A}_{\mathrm{rd}}$.

2.4 Congestion cost

Notice that the arc flows $f_{a_{\mathrm{sd}}}^{\mathrm{am/pm}}$, $f_{a_{\mathrm{rd}}}^{\mathrm{am/pm}}$, and $f_{a_{\mathrm{hp}}}^{\mathrm{am/pm}}$, representing the number of solo driving vehicles, the rideshare vehicles, and ride-hailing vehicles, respectively, are the sources of traffic congestion; but $f_{a_{\mathrm{rp}}}^{\mathrm{am/pm}}$ are not (rideshare passenger flows do not influence traffic congestion). Let $\boldsymbol{f}^{\mathrm{am}} \triangleq \{ f_a^{\mathrm{am}} \}_{a \in \mathcal{A}}$ and $\boldsymbol{f}^{\mathrm{pm}} \triangleq \{ f_a^{\mathrm{pm}} \}_{a \in \mathcal{A}}$ where $\mathcal{A} = \mathcal{A}_{\mathrm{sd}} \cup \mathcal{A}_{\mathrm{rd}} \cup \mathcal{A}_{\mathrm{rp}} \cup \mathcal{A}_{\mathrm{hp}}$. Derived from the Bureau of Public Roads (BPR) functions, we obtain the travel time $\mathrm{tt}_a(\bullet)$ as follows

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$$\left.\begin{aligned} \mathrm{tt}_a^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) &= t_a\left(1+b\left(\frac{f_{a_{\mathrm{sd}}}^{\mathrm{am}}+f_{a_{\mathrm{rd}}}^{\mathrm{am}}+f_{a_{\mathrm{hp}}}^{\mathrm{am}}}{\alpha_a}\right)^4\right) \\ \mathrm{tt}_a^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) &= t_a\left(1+b\left(\frac{f_{a_{\mathrm{sd}}}^{\mathrm{pm}}+f_{a_{\mathrm{rd}}}^{\mathrm{pm}}+f_{a_{\mathrm{hp}}}^{\mathrm{pm}}}{\alpha_a}\right)^4\right) \end{aligned}\right\} \quad \forall\, a \in \mathcal{A}_0. \tag{1}$$

where $a_{\mathrm{sd}} = \mathcal{T}_{\mathrm{sd}}(a)$, $a_{\mathrm{rd}} = \mathcal{T}_{\mathrm{rd}}(a)$, and $a_{\mathrm{hp}} = \mathcal{T}_{\mathrm{hp}}(a)$ for $a \in \mathcal{A}_0$ are the corresponding arcs for solo drivers, ridesharing drivers, and ride-hailing vehicles respectively, splitted from the original arc $a \in \mathcal{A}_0$; t_a and α_a represent the free flow travel time and flow capacity of arc $a \in \mathcal{A}_0$, respectively.

2.5 Inconvenience cost and payment/income

In this section, we define the inconvenience costs and payments/incomes of rideshare and ride-hailing services. The cost structure is similar to that of existing rideshare user equilibrium literature (e.g., Xu et al., 2015b; Ma et al., 2020; Li et al., 2020; Ma et al., 2022). To include ride-hailing services in our general traffic equilibrium framework, we also formulate the inconvenience cost and payment for ride-hailing passengers.

- *Inconvenience cost of rideshare drivers*: In addition to the congestion cost in Equation (1), a rideshare driver will also experience the inconvenience for sharing the vehicle with passengers, which includes but is not limited to picking up, dropping off, or even waiting for passengers. The inconvenience cost of rideshare drivers is defined as follows

$$\left.\begin{aligned} &I_{a;\mathrm{rd}}^{\mathrm{am}}\left(f_{a_{\mathrm{rp}}}^{\mathrm{am}}\right) \\ &I_{a;\mathrm{rd}}^{\mathrm{pm}}\left(f_{a_{\mathrm{rp}}}^{\mathrm{pm}}\right) \end{aligned}\right\} \quad \forall\, a \in \mathcal{A}_{\mathrm{rd}} \text{ with } a_{\mathrm{rp}} = \mathcal{T}_{\mathrm{rp}}\left(\mathcal{T}_0\left(a\right)\right) \tag{2}$$

where $I_{a;\mathrm{rd}}^{\mathrm{am}}(\bullet)$ and $I_{a;\mathrm{rd}}^{\mathrm{pm}}(\bullet)$ are monotone increasing functions, namely the inconvenience cost for rideshare drivers will increase when there are more rideshare passengers. This is because when there are more rideshare passengers, the rideshare drivers will need more detours for picking up or dropping off passengers, which leads to higher inconvenience cost for rideshare drivers.

- *Inconvenience cost of rideshare passengers*: Similar to rideshare drivers, the rideshare service could also cause some inconvenience for rideshare passengers. The inconvenience cost may include the waiting time for drivers to pick them up, possible detour together with the drivers for picking up or dropping off other passengers, or even the anxiety to share a ride with strangers. The inconvenience cost of rideshare passengers is given by

$$\left.\begin{array}{l} I_{a;\mathrm{rp}}^{\mathrm{am}}\left(f_a^{\mathrm{am}}\right) \\ \\ I_{a;\mathrm{rp}}^{\mathrm{pm}}\left(f_a^{\mathrm{pm}}\right) \end{array}\right\} \quad \forall\, a \in \mathcal{A}_{\mathrm{rp}} \tag{3}$$

where $I_{a;\mathrm{rp}}^{\mathrm{am}}(\bullet)$ and $I_{a;\mathrm{rp}}^{\mathrm{pm}}(\bullet)$ are monotone increasing functions. When there are more rideshare passengers in the system, the rideshare passengers will need longer waiting time for rideshare drivers to pick them up and will possibly suffer more discomfort for sharing the ride, which leads to higher inconvenience cost for rideshare passengers.

- *Inconvenience cost of ride-hailing passengers*: Similar to rideshare passengers, ride-hailing passengers also experience some inconvenience, which may come from waiting for picking up after calling a ride. The inconvenience cost of ride-hailing passengers is as follows

$$\left.\begin{array}{l} I_{a;\mathrm{hp}}^{\mathrm{am}}\left(f_a^{\mathrm{am}}\right) \\ \\ I_{a;\mathrm{hp}}^{\mathrm{pm}}\left(f_a^{\mathrm{pm}}\right) \end{array}\right\} \quad \forall\, a \in \mathcal{A}_{\mathrm{hp}} \tag{4}$$

where $I_{a;\mathrm{hp}}^{\mathrm{am}}(\bullet)$ and $I_{a;\mathrm{hp}}^{\mathrm{pm}}(\bullet)$ are monotone increasing functions, namely the inconvenience cost of ride-hailing passengers increases as there are more ride-hailing passengers. The reason is that when there are more ride-hailing passengers in the system, there is more demand for this mode type, and as a result, the ride-hailing passengers experience longer waiting time, resulting in larger inconvenience cost.

- *Payment of rideshare passengers*: The main reason that the rideshare drivers may be willing to provide rideshare service and may want to pick up more passengers is that they can receive compensation to cover part of their driving cost from each of the rideshare passengers. The payment of each rideshare passenger, which may be different in the morning and evening, is defined as

$$\left.\begin{array}{lcl} R_{a;\mathrm{rp}}^{\mathrm{am}}\left(\boldsymbol{f}^{\mathrm{am}}\right) & \triangleq & B_{a;\mathrm{rp}}^{\mathrm{am}} - S_{a;\mathrm{rp}}^{\mathrm{am}}(f_{a_{\mathrm{rd}}}^{\mathrm{am}}) + E_{a;\mathrm{rp}}^{\mathrm{am}}(f_a^{\mathrm{am}}) \\ \\ R_{a;\mathrm{rp}}^{\mathrm{pm}}\left(\boldsymbol{f}^{\mathrm{pm}}\right) & \triangleq & B_{a;\mathrm{rp}}^{\mathrm{pm}} - S_{a;\mathrm{rp}}^{\mathrm{pm}}(f_{a_{\mathrm{rd}}}^{\mathrm{pm}}) + E_{a;\mathrm{rp}}^{\mathrm{pm}}(f_a^{\mathrm{pm}}) \end{array}\right\} \quad \begin{array}{l} \forall\, a \in \mathcal{A}_{\mathrm{rp}} \text{ with } a_0 = \mathcal{T}_0(a), \\ \text{and } a_{\mathrm{rd}} = \mathcal{T}_{\mathrm{rd}}\left(\mathcal{T}_0\left(a\right)\right) \end{array} \tag{5}$$

where $B_{a;\mathrm{rp}}^{\mathrm{am}}$ and $B_{a;\mathrm{rp}}^{\mathrm{pm}}$ are positive constants; $S_{a;\mathrm{rp}}^{\mathrm{am}}(\bullet)$, $S_{a;\mathrm{rp}}^{\mathrm{pm}}(\bullet)$, $E_{a;\mathrm{rp}}^{\mathrm{am}}(\bullet)$ and $E_{a;\mathrm{rp}}^{\mathrm{pm}}(\bullet)$ are monotone increasing functions. The first part of Equation (5) is the benchmark price of rideshare passengers for arc $a \in \mathcal{A}_{\mathrm{rp}}$. The second and third part of Equation (5) are related to the relationship between supply and demand of the rideshare market: when there are more rideshare drivers, i.e., the supply of the rideshare market is larger, the price for rideshare service will decrease; when there are more rideshare passengers, namely the demand of the rideshare market becomes larger, the payment for rideshare service will increase.

- *Income of rideshare drivers*: Rideshare driver's income is equal to the summation of all payments of rideshare passengers in his/her car. Since the income of rideshare drivers is defined at the OD level, the actual number of passengers per vehicle for each OD pair is not predetermined, but should be within the range $[1, M]$. That is to say, on average there will be at least one passenger and at most M passengers in each vehicle for each OD pair. Similar to Xu et al. (2015b), for simplicity we set the average number of rideshare passengers per rideshare vehicle for each OD pair to be fixed constants $\kappa^{\mathrm{am}}, \kappa^{\mathrm{pm}} \in [1, M]$, namely,

$$\left.\begin{aligned} R^{\mathrm{am}}_{a;\mathrm{rd}}\left(\boldsymbol{f}^{\,\mathrm{am}}\right) &\triangleq \kappa^{\,\mathrm{am}} R^{\mathrm{am}}_{a_{\mathrm{rp}};\mathrm{rp}}\left(\boldsymbol{f}^{\,\mathrm{am}}\right) \\ R^{\mathrm{pm}}_{a;\mathrm{rd}}\left(\boldsymbol{f}^{\,\mathrm{pm}}\right) &\triangleq \kappa^{\,\mathrm{pm}} R^{\mathrm{pm}}_{a_{\mathrm{rp}};\mathrm{rp}}\left(\boldsymbol{f}^{\,\mathrm{pm}}\right) \end{aligned}\right\} \quad \forall\, a \in \mathcal{A}_{\mathrm{rd}} \text{ with } a_{\mathrm{rp}} = \mathcal{T}_{\mathrm{rp}}\left(\mathcal{T}_0\left(a\right)\right) \tag{6}$$

- *Payment of ride-hailing passengers*: Similar to rideshare passengers, ride-hailing passengers also need to pay for using the ride-hailing service. The payment for each ride-hailing passenger is

$$\left.\begin{aligned} R^{\mathrm{am}}_{a;\mathrm{hp}}\left(\boldsymbol{f}^{\,\mathrm{am}}\right) &\triangleq B^{\mathrm{am}}_{a;\mathrm{hp}} + E^{\mathrm{am}}_{a;\mathrm{hp}}(f^{\mathrm{am}}_a) \\ R^{\mathrm{pm}}_{a;\mathrm{hp}}\left(\boldsymbol{f}^{\,\mathrm{pm}}\right) &\triangleq B^{\mathrm{pm}}_{a;\mathrm{hp}} + E^{\mathrm{pm}}_{a;\mathrm{hp}}(f^{\mathrm{pm}}_a) \end{aligned}\right\} \quad \forall\, a \in \mathcal{A}_{\mathrm{hp}} \text{ with } a_0 = \mathcal{T}_0(a) \tag{7}$$

where $B^{\mathrm{am}}_{a;\mathrm{hp}}$ and $B^{\mathrm{pm}}_{a;\mathrm{hp}}$ are positive constants that represent the benchmark prices of ride-hailing passengers for arc $a \in \mathcal{A}_{\mathrm{hp}}$; $E^{\mathrm{am}}_{a;\mathrm{hp}}(\bullet)$ and $E^{\mathrm{pm}}_{a;\mathrm{hp}}(\bullet)$ are monotone increasing functions.

2.6 The overall arc and path cost functions

Denote ψ as the conversion factor of time (minutes) to money (dollars). Based on Section 2.4 and 2.5, in sum, each traveler on arc $a \in \mathcal{A}$ experiences a total cost of

$$\begin{aligned}
&\mathrm{tc}_a^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) = \\
&\begin{cases}
\psi \times \mathrm{tt}_{a_0}^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) & \forall a \in \mathcal{A}_{\mathrm{sd}} \text{ with } a_0 = \mathcal{T}_0(a) \\
\psi \times \mathrm{tt}_{a_0}^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) + I_{a;\mathrm{rd}}^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) - R_{a;\mathrm{rd}}^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) & \forall a \in \mathcal{A}_{\mathrm{rd}} \text{ with } a_0 = \mathcal{T}_0(a) \\
\psi \times \mathrm{tt}_{a_0}^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) + I_{a;\mathrm{rp}}^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) + R_{a;\mathrm{rp}}^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) & \forall a \in \mathcal{A}_{\mathrm{rp}} \text{ with } a_0 = \mathcal{T}_0(a) \\
\psi \times \mathrm{tt}_{a_0}^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) + I_{a;\mathrm{hp}}^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) + R_{a;\mathrm{hp}}^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) & \forall a \in \mathcal{A}_{\mathrm{hp}} \text{ with } a_0 = \mathcal{T}_0(a)
\end{cases} \\
&\mathrm{tc}_a^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) = \\
&\begin{cases}
\psi \times \mathrm{tt}_{a_0}^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) & \forall a \in \mathcal{A}_{\mathrm{sd}} \text{ with } a_0 = \mathcal{T}_0(a) \\
\psi \times \mathrm{tt}_{a_0}^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) + I_{a;\mathrm{rd}}^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) - R_{a;\mathrm{rd}}^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) & \forall a \in \mathcal{A}_{\mathrm{rd}} \text{ with } a_0 = \mathcal{T}_0(a) \\
\psi \times \mathrm{tt}_{a_0}^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) + I_{a;\mathrm{rp}}^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) + R_{a;\mathrm{rp}}^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) & \forall a \in \mathcal{A}_{\mathrm{rp}} \text{ with } a_0 = \mathcal{T}_0(a) \\
\psi \times \mathrm{tt}_{a_0}^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) + I_{a;\mathrm{hp}}^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) + R_{a;\mathrm{hp}}^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}) & \forall a \in \mathcal{A}_{\mathrm{hp}} \text{ with } a_0 = \mathcal{T}_0(a)
\end{cases}
\end{aligned} \tag{8}$$

Denote z_p as the flow of travelers of path $p \in \mathcal{P}_k$ joining OD pair $k \in \mathcal{K}$ through the entire day; denote $\delta_{a;p}^{\mathrm{am}}$ and $\delta_{a;p}^{\mathrm{pm}}$ as the arc-path indicators in the morning and in the evening, respectively. Let $\boldsymbol{z} \triangleq \{z_p\}_{p \in \mathcal{P}}$, $\boldsymbol{\Delta}^{\mathrm{am}} \triangleq (\delta_{a;p}^{\mathrm{am}})_{(a,p) \in \mathcal{A} \times \mathcal{P}}$ and $\boldsymbol{\Delta}^{\mathrm{pm}} \triangleq (\delta_{a;p}^{\mathrm{pm}})_{(a,p) \in \mathcal{A} \times \mathcal{P}}$. Similar to the path cost structure of Xu et al. (2015b) and Li et al. (2020), the total cost experienced by a traveler throughout the entire day on path $p \in \mathcal{P}$ can be represented as

$$\begin{aligned}
\mathrm{TC}_p(\boldsymbol{z}) &= \sum_{a \in \mathcal{A}} \left[\delta_{a;p}^{\mathrm{am}} \times \mathrm{tc}_a^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}) + \delta_{a;p}^{\mathrm{pm}} \times \mathrm{tc}_a^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}})\right] \\
&= (\boldsymbol{\Delta}^{\mathrm{am}})^T \mathrm{tc}_a^{\mathrm{am}}(\boldsymbol{\Delta}^{\mathrm{am}} \boldsymbol{z}) + (\boldsymbol{\Delta}^{\mathrm{pm}})^T \mathrm{tc}_a^{\mathrm{pm}}(\boldsymbol{\Delta}^{\mathrm{pm}} \boldsymbol{z})
\end{aligned} \tag{9}$$

2.7 Rideshare capacity constraints

Denote the passenger capacity of each rideshare vehicle as M. We have the rideshare capacity constraints as follows:

$$\left\{\begin{array}{l} f_{a_{\text{rd}}}^{\text{am}} \leq f_a^{\text{am}} \leq M \times f_{a_{\text{rd}}}^{\text{am}} \\ \\ f_{a_{\text{rd}}}^{\text{pm}} \leq f_a^{\text{pm}} \leq M \times f_{a_{\text{rd}}}^{\text{pm}} \end{array}\right\} \forall\, a \in \mathcal{A}_{\text{rp}} \text{ with } a_{\text{rd}} = \mathcal{T}_{\text{rd}}\left(\mathcal{T}_0\left(a\right)\right) \tag{10}$$

namely

$$\left.\begin{array}{l} \displaystyle\sum_{p\in\mathcal{P},\, a_{\text{rd}}\in p} z_p \leq \sum_{p\in\mathcal{P},\, a_{\text{rp}}\in p} z_p \leq M \sum_{p\in\mathcal{P},\, a_{\text{rd}}\in p} z_p \\ \displaystyle\sum_{p\in\mathcal{P},\, a_{\text{rd}}\in p} z_p \leq \sum_{p\in\mathcal{P},\, a_{\text{rp}}\in p} z_p \leq M \sum_{p\in\mathcal{P},\, a_{\text{rd}}\in c} z_p \end{array}\right\} \begin{array}{l} \forall\, a \in \mathcal{A}_0, a_{\text{rd}} = \mathcal{T}_{\text{rd}}(a), \\ \text{and } a_{\text{rp}} = \mathcal{T}_{\text{rp}}(a) \end{array} \tag{11}$$

These constraints ensure two things on each arc: (i) the total number of rideshare drivers does not exceed the total number of rideshare passengers; (ii) the total number of rideshare passengers is no more than M times the total number of rideshare drivers due to the vehicular capacity.

Denote $\perp$ as the perpendicularity notation, $\mathrm{x} \perp \mathrm{y} \iff \mathrm{x}^T\mathrm{y} = 0$ (see Definition 1.1.5 of Facchinei and Pang (2003)). Then the equivalent complementarity formulation of the Inequilities (10) can be written as follows:

$$\left.\begin{array}{l} 0 \leq \eta_a^{+;\text{am}} \perp f_{a_{\text{rp}}}^{\text{am}} - f_{a_{\text{rd}}}^{\text{am}} \geq 0 \\ 0 \leq \eta_a^{-;\text{am}} \perp M\, f_{a_{\text{rd}}}^{\text{am}} - f_{a_{\text{rp}}}^{\text{am}} \geq 0 \\ 0 \leq \eta_a^{+;\text{pm}} \perp f_{a_{\text{rp}}}^{\text{pm}} - f_{a_{\text{rd}}}^{\text{pm}} \geq 0 \\ 0 \leq \eta_a^{-;\text{pm}} \perp M\, f_{a_{\text{rd}}}^{\text{pm}} - f_{a_{\text{rp}}}^{\text{pm}} \geq 0 \end{array}\right\} \begin{array}{l} \forall\, a \in \mathcal{A}_0, \\ a_{\text{rd}} = \mathcal{T}_{\text{rd}}(a), \\ \text{and } a_{\text{rp}} = \mathcal{T}_{\text{rp}}(a) \end{array} \tag{12}$$

where the variables $\eta_a^{\pm;\text{am}}$ and $\eta_a^{\pm;\text{pm}}$ are the shadow prices to help enforce the morning and evening rideshare capacity constraints, respectively. For example, only when $f_{a_{\text{rd}}}^{\text{am}} = f_{a_{\text{rp}}}^{\text{am}}$ for some $a_{\text{rd}} = \mathcal{T}_{\text{rd}}(a)$, $a_{\text{rp}} = \mathcal{T}_{\text{rp}}(a)$ and $a \in \mathcal{A}_0$, namely the flow on rideshare-driver arc equals to the relevant flow on rideshare-passenger arc, there could be a compensation $\eta_a^{+;\text{am}} > 0$ incurred to avoid the situation that $f_{a_{\text{rd}}}^{\text{am}} > f_{a_{\text{rp}}}^{\text{am}}$; similarly, only when $M \times f_{a_{\text{rd}}}^{\text{am}} = f_{a_{\text{rp}}}^{\text{am}}$ for some $a_{\text{rd}} = \mathcal{T}_{\text{rd}}(a)$, $a_{\text{rp}} = \mathcal{T}_{\text{rp}}(a)$ and $a \in \mathcal{A}_0$, namely the rideshare-passenger arc is at capacity, then there could be an extra payment $\eta_a^{-;\text{am}} > 0$ incurred to prevent the situation that $f_{a_{\text{rp}}}^{\text{am}} > M \times f_{a_{\text{rd}}}^{\text{am}}$.

2.8 Demand satisfaction and flow conservation equations

Demand satisfaction equations are used to balance total trip demands with path flows and ensuring morning trip demands equal evening trip demands, namely morning and evening trip demands are aggregated to total trip demands:

$$D_k = \sum_{p \in \mathcal{P}_k} z_p \quad \forall\, k \in \mathcal{K} \tag{13}$$

The above path-based demand satisfaction equations (13) can be reformulated by arc flows, but extra flow conservation equations will be needed. First, we decompose the morning arc flows f_a^{am} and evening arc flows f_a^{pm} by morning OD pair $k \in \mathcal{K}$ and evening return OD pair $\bar{k}$, respectively. Denote $x_{a;\,k}^{\text{am}} \geq 0$ and $x_{a;\,\bar{k}}^{\text{pm}} \geq 0$ as the amount of flow for OD pair $k \in \mathcal{K}$ on arc $a \in \mathcal{A}$ in the morning and the amount of flow for associated return OD pair $\bar{k}$ on arc $a \in \mathcal{A}$ in the evening, respectively. Then we have the arc flow decomposition as follows:

$$f_a^{\text{am}} = \sum_{k \in \mathcal{K}} x_{a;\,k}^{\text{am}} \quad \text{and} \quad f_a^{\text{pm}} = \sum_{\bar{k} \in \bar{\mathcal{K}}} x_{a;\,\bar{k}}^{\text{pm}} \quad \forall\, a \in \mathcal{A} \tag{14}$$

Demand satisfaction equations can then be represented from arc perspective as follows:

$$\begin{aligned} &\sum_{a \in \text{IN}(d_k)} x_{a;\,k}^{\text{am}} = \sum_{a \in \text{OUT}(o_k)} x_{a;\,k}^{\text{am}} = \sum_{a \in \text{IN}(d_{\bar{k}})} x_{a;\,\bar{k}}^{\text{pm}} = \sum_{a \in \text{OUT}(o_{\bar{k}})} x_{a;\,\bar{k}}^{\text{pm}} = D_k \\ &\forall k \in \mathcal{K} \text{ and associated } \bar{k} \end{aligned} \tag{15}$$

where o_k and d_k represent the origin and destination of OD pair k in the morning; $o_{\bar{k}}$ and $d_{\bar{k}}$ represent the origin and destination of the associated return OD pair $\bar{k}$ in the evening; $\text{IN}(i)$ and $\text{OUT}(i)$ represent the sets of arcs entering and leaving node $i \in \mathcal{N}$, respectively.

We also need the flow conservation equations below for the nodes other than origins and destinations. The flow conservation equations guarantee the inflow of a node $i \in \mathcal{N}$ is equal to the outflow of that node, which can be formulated as follows:

$$\left\{ \begin{aligned} &\sum_{a \in \text{IN}(i)} x_{a;\,k}^{\text{am}} - \sum_{a \in \text{OUT}(i)} x_{a;\,k}^{\text{am}} = 0 \quad \forall i \in \mathcal{N} \setminus \{o_k, d_k\}, \forall k \in \mathcal{K} \\ &\sum_{a \in \text{IN}(i)} x_{a;\,\bar{k}}^{\text{pm}} - \sum_{a \in \text{OUT}(i)} x_{a;\,\bar{k}}^{\text{pm}} = 0 \quad \forall i \in \mathcal{N} \setminus \{o_{\bar{k}}, d_{\bar{k}}\}, \text{ associated } \bar{k} \end{aligned} \right\} \tag{16}$$

In order to maintain the formulation a square system (i.e., number of variables equal to number of constraints), we assume that the morning drivers will remain to be drivers in the evening. As a result, total number of morning and evening drivers should be equal:

$$\sum_{a_{\mathrm{sd}}, a_{\mathrm{rd}} \in \mathrm{OUT}(o_k)} (x^{\mathrm{am}}_{a_{\mathrm{sd}};\, k} + x^{\mathrm{am}}_{a_{\mathrm{rd}};\, k}) = \sum_{a_{\mathrm{sd}}, a_{\mathrm{rd}} \in \mathrm{IN}(d_{\bar{k}})} (x^{\mathrm{pm}}_{a_{\mathrm{sd}};\, \bar{k}} + x^{\mathrm{pm}}_{a_{\mathrm{rd}};\, \bar{k}}) \quad \forall k \in \mathcal{K} \text{ and associated } \bar{k} \tag{17}$$

The demand satisfaction equations (15) and the driver flow conservation equations (17) together guarantee that the total number of morning and evening passengers are equal. This is consistent with the fact that if a traveler chooses to be a (rideshare or ride-hailing) passenger in the morning, (s)he will have to take a ride back home in the evening.

2.9 The overall equilibrium model

In this section, we summarize the aforementioned sections and develop a general equilibrium model to capture the complicated interactions between solo drivers, rideshare drivers, rideshare passengers and ride-hailing passengers that allows travelers to switch between different transportation modes in a coupled morning-evening commute framework.

Based on path flows and path cost functions, the model is formulated as a variational inequality (VI) defined by the pair of mapping Φ and feasible set $\mathcal{HF}$, notated as $\mathrm{VI}(\Phi, \mathcal{HF})$, as follows:

$$\begin{aligned} \Phi(\boldsymbol{z}) &\triangleq \left(\, (\,\mathrm{TC}_p(\boldsymbol{z})\,)_{p \in \mathcal{P}} \, \right), \\ \mathcal{HF} &\triangleq \left\{ \, \boldsymbol{z} \triangleq (\, z_p \,)_{p \in \mathcal{P}} \geq 0 \;\middle|\; \text{subject to } (11), (13) \, \right\}. \end{aligned}$$

Similarly, based on arc flows and arc cost functions, the model is formulated as a VI defined by the pair of mapping Φ' and feasible set $\mathcal{HF}'$, notated as $\mathrm{VI}(\Phi', \mathcal{HF}')$, as follows:

$$\Phi'(\boldsymbol{f}) \triangleq \left((\mathrm{tc}_a^{\mathrm{am}}(\boldsymbol{f}^{\mathrm{am}}),\, \mathrm{tc}_a^{\mathrm{pm}}(\boldsymbol{f}^{\mathrm{pm}}))_{a\in\mathcal{A}} \right),$$

$$\mathcal{HF}' \triangleq \left\{ \boldsymbol{f} \triangleq \left(f_a^{\mathrm{am/pm}} \right)_{a\in\mathcal{A}} \;\middle|\; \begin{array}{l} \exists\, \boldsymbol{x}_k^{\mathrm{am}} \triangleq \left\{ x_{a;\,k}^{\mathrm{am}} \right\}_{a\in\mathcal{A}} \geq 0 \text{ and} \\ \boldsymbol{x}_{\bar{k}}^{\mathrm{pm}} \triangleq \left\{ x_{a;\,\bar{k}}^{\mathrm{pm}} \right\}_{a\in\mathcal{A}} \geq 0 \text{ satisfying} \\ (10),\ (14),\ (15),\ (16),\ (17) \end{array} \right\}.$$

Remarks:

(i) By Definition 1.1.1 of Facchinei and Pang (2003), variational inequality VI($\Phi, \mathcal{HF}$) is the problem to find a vector $\boldsymbol{z} \in \mathcal{HF}$ such that $(\bar{\boldsymbol{z}} - \boldsymbol{z})^T \Phi(\boldsymbol{z}) \geq 0 \quad \forall \bar{\boldsymbol{z}} \in \mathcal{HF}$;

(ii) The path-based VI($\Phi, \mathcal{HF}$) and the arc-based VI($\Phi', \mathcal{HF}'$) are equivalent. The proof is as follows:

$$\begin{aligned} & (\bar{\boldsymbol{f}} - \boldsymbol{f})^T \Phi'(\boldsymbol{f}) \geq 0 \quad \forall \bar{\boldsymbol{f}} \in \mathcal{HF}' \\ \Longleftrightarrow\ & (\boldsymbol{\Delta}\bar{\boldsymbol{z}} - \boldsymbol{\Delta}\boldsymbol{z})^T \Phi'(\boldsymbol{\Delta}\boldsymbol{z}) \geq 0 \quad \forall \bar{\boldsymbol{z}} \in \mathcal{HF} \\ \Longleftrightarrow\ & (\bar{\boldsymbol{z}} - \boldsymbol{z})^T \boldsymbol{\Delta}^T \Phi'(\boldsymbol{\Delta}\boldsymbol{z}) \geq 0 \quad \forall \bar{\boldsymbol{z}} \in \mathcal{HF} \\ \Longleftrightarrow\ & (\bar{\boldsymbol{z}} - \boldsymbol{z})^T \Phi(\boldsymbol{z}) \geq 0 \quad \forall \bar{\boldsymbol{z}} \in \mathcal{HF} \end{aligned}$$

where $\boldsymbol{\Delta} \triangleq (\delta_{a;p})_{(a,p)\in\mathcal{A}\times\mathcal{P}}$ and $\delta_{a;p}$ are the arc-path indicators;

(iii) Since the mapping Φ is continuous and the set $\mathcal{HF}$ is compact and convex, by Corollary 2.2.5 of Facchinei and Pang (2003) it follows that the path-based VI($\Phi, \mathcal{HF}$) has a solution. Similarly, we can also show that the arc-based VI($\Phi', \mathcal{HF}'$) has a solution. Thus so does our coupled morning-evening commute model.

2.10 The extended user equilibrium conditions

We proposed an extended user equilibrium principle that describes a complementary relation between the daily commute path flows and the travelers' minimal disutiltities; it is based on the combined morning-evening round trips, allowing the switches of commute types. This type of equilibrium distinguishes itself from the separate morning equilibrium and evening equilibrium. The disutilities pertain to each OD pair $k \in \mathcal{K}$ and path flows z_p of all the morning-evening paths $p \in \mathcal{P}_k$ in the extended network connecting that OD pair. That is to say, for each OD pair k, the chosen morning-evening paths $p \in \mathcal{P}_k$ connecting this OD pair among the 4 travel modes in Fig. 1 will all have travel costs equal to minimum travel cost for all paths of the OD pair in question, and this common cost does not exceed the travel costs of the unchosen morning-evening paths for the travel mode joining the same OD pair. This extends Wardrop's user equilibrium principle for the coupled morning-evening commute instead of the separate morning or evening commute in a traditional traffic equilibrium problem.

Another difference compared to the traditional user equilibrium is that travelers' total cost includes not only the path-based cost defined in Section 2.6, but also the compensations induced by the rideshare capacity constraints in Section 2.7. Denote u_k as the generalized (least) disutility of OD pair $k \in \mathcal{K}$, which is the minimum generalized travel cost under an equilibrium state for all paths connecting OD pair $k \in \mathcal{K}$. Written as the equivalent complementarity formulation of path-based cost function (9), the extended path-based user equilibrium conditions for the combined morning and evening commutes among the 4 types of travel modes in Fig. 1 are:

$$0 \leq z_p \perp \mathrm{TC}_p(\boldsymbol{z}) - \underbrace{\lambda_p(\boldsymbol{\eta}_a)}_{\text{compensations}} - \underbrace{u_k}_{\text{least disutility}} \geq 0, \quad \forall\, p \in \mathcal{P} \tag{18}$$

where, in this context, the perpendicularity notation $\perp$ asserts the complementarity between the morning-evening path flows and the travelers' deviations from the minimal disutilities. In other words, if a traveler chooses the morning-evening path $p \in \mathcal{P}$, then the path cost/disutility must be the minimum of all costs for this OD pair k. Denote $p^{\,\mathrm{am}} \in \mathcal{P}^{\,\mathrm{am}}$ as the morning path of p, $p^{\,\mathrm{pm}} \in \mathcal{P}^{\,\mathrm{pm}}$ as the evening path of p. Let $\boldsymbol{\eta_a} \triangleq (\eta_a^{+;\mathrm{am}}, \eta_a^{-;\mathrm{am}}, \eta_a^{+;\mathrm{pm}}, \eta_a^{-;\mathrm{pm}})_{a \in \mathcal{A}_0}$, here for all $p \in \mathcal{P} = \mathcal{P}^{\,\mathrm{am}} \times \mathcal{P}^{\,\mathrm{pm}}$ we have that,

$$\begin{aligned} \lambda_p(\boldsymbol{\eta}_a) \;\triangleq\; & \sum_{\mathcal{T}_{\mathrm{rd}}(a) \in p^{\,\mathrm{am}} \cap \mathcal{A}_{\mathrm{rd}}} \left(M\, \eta_a^{-;\mathrm{am}} - \eta_a^{+;\mathrm{am}} \right) + \sum_{\mathcal{T}_{\mathrm{rd}}(a) \in p^{\,\mathrm{pm}} \cap \mathcal{A}_{\mathrm{rd}}} \left(M\, \eta_a^{-;\mathrm{pm}} - \eta_a^{+;\mathrm{pm}} \right) \\ & + \sum_{\mathcal{T}_{\mathrm{rp}}(a) \in p^{\,\mathrm{am}} \cap \mathcal{A}_{\mathrm{rp}}} \left(\eta_a^{+;\mathrm{am}} - \eta_a^{-;\mathrm{am}} \right) + \sum_{\mathcal{T}_{\mathrm{rp}}(a) \in p^{\,\mathrm{pm}} \cap \mathcal{A}_{\mathrm{rp}}} \left(\eta_a^{+;\mathrm{pm}} - \eta_a^{-;\mathrm{pm}} \right) \end{aligned}$$

The extended user equilibrium conditions can also be formulated from an arc perspective. That is to say, for each OD pair k, the chosen arc $a \in \mathcal{A}$ in the extended network connecting this OD pair among the 4 travel modes in Fig. 1 will all have travel costs equal to the least disutility of the

OD pair in question, and this common cost does not exceed the travel costs of the unchosen arcs for the travel mode joining the same OD pair. One difference compared to the extended path-based user equilibrium is that travelers' total cost includes not only the arc-based costs defined in Section 2.6, the compensations ($\eta_a^{\pm;\text{am}}$, $\eta_a^{\pm;\text{pm}}$) induced by the rideshare capacity constraints in Section 2.7, but also the multipliers ($\mu_i^{k;\text{am}}$, $\mu_i^{\bar{k};\text{pm}}$, $\zeta_{a;\,k}$) induced by demand satisfaction and flow conservation equations in Section 2.8. Thus the extended arc-based user equilibrium conditions for the combined morning and evening commutes are:

$$
\begin{aligned}
&0 \le x_{a;\,k}^{\text{am}} \perp \text{tc}_a^{\text{am}}(\boldsymbol{x}_k^{\text{am}}) + \underbrace{\omega_a^{+;\text{am}}\eta_{\mathcal{T}_0(a)}^{+;\text{am}} + \omega_a^{-;\text{am}}\eta_{\mathcal{T}_0(a)}^{-;\text{am}}}_{\text{compensations}} \underbrace{-\mu_i^{k;\text{am}} + \mu_j^{k;\text{am}} - \omega_a^{\text{am}}\zeta_{a;\,k}}_{\text{least disutility}} \ge 0,\\
&\forall a = (i,j) \in \mathcal{A},\ \forall k \in \mathcal{K}\\
&0 \le x_{a;\,\bar{k}}^{\text{pm}} \perp \text{tc}_a^{\text{pm}}(\boldsymbol{x}_{\bar{k}}^{\text{pm}}) + \underbrace{\omega_a^{+;\text{pm}}\eta_{\mathcal{T}_0(a)}^{+;\text{pm}} + \omega_a^{-;\text{pm}}\eta_{\mathcal{T}_0(a)}^{-;\text{pm}}}_{\text{compensations}} \underbrace{-\mu_i^{\bar{k};\text{pm}} + \mu_j^{\bar{k};\text{pm}} + \omega_a^{\text{pm}}\zeta_{a;\,k}}_{\text{least disutility}} \ge 0,\\
&\forall a = (i,j) \in \mathcal{A},\ \text{associated } \bar{k}
\end{aligned}
\tag{19}
$$

where

$$
\omega_a^{+;\text{am}} = \omega_a^{+;\text{pm}} \triangleq \begin{cases} 0, \text{ if } a \in \mathcal{A}_{\text{sd}} \cup \mathcal{A}_{\text{hp}} \\ 1, \text{ if } a \in \mathcal{A}_{\text{rd}} \\ -1, \text{ if } a \in \mathcal{A}_{\text{rp}} \end{cases} \quad \text{and} \quad \omega_a^{-;\text{am}} = \omega_a^{-;\text{pm}} \triangleq \begin{cases} 0, \text{ if } a \in \mathcal{A}_{\text{sd}} \cup \mathcal{A}_{\text{hp}} \\ -M, \text{ if } a \in \mathcal{A}_{\text{rd}} \\ 1, \text{ if } a \in \mathcal{A}_{\text{rp}} \end{cases}
$$

$$
\omega_a^{\text{am}} = \omega_a^{\text{pm}} \triangleq \begin{cases} 1, \text{ if } a \in \mathcal{A}_{\text{sd}} \cup \mathcal{A}_{\text{rd}} \\ 0, \text{ if } a \in \mathcal{A}_{\text{rp}} \cup \mathcal{A}_{\text{hp}} \end{cases}
$$

Although the extended arc-based user equilibrium conditions (19) do not appear intuitive, it is equivalent to the extended path-based user equilibrium conditions (18). This comes from the equivalence of the path-based VI($\Phi, \mathcal{HF}$) and the arc-based VI($\Phi', \mathcal{HF}'$), which has been shown in remark (ii) of Section 2.9. Since the feasible sets $\mathcal{HF}$ and $\mathcal{HF}'$ are polyhedra, their equivalent mixed complementarity problem (MiCP) formulations derived from Proposition 1.2.1 of Facchinei and Pang (2003) must also be equivalent. For the overall equivalent MiCP of the arc-based VI model, please refer to Appendix 1, which provides details about how the multipliers ($\mu_i^{k;\text{am}}$, $\mu_i^{\bar{k};\text{pm}}$, $\zeta_{a;\,k}$) are induced.

3 Uniqueness of The Equilibrium

Existence of an equilibrium is shown in remark (iii) of Section 2.9. In this section, we derive the conditions under which our proposed model will have a unique solution. We provide the condition on the model parameters under which the equilibrium will be globally unique. Furthermore, we show that, under the same condition, the equilibrium will be locally unique even when a commonly used assumption in the literature is violated.

We analyze the Jacobian matrix $J\Phi(\boldsymbol{f}^{\,\mathrm{am}}, \boldsymbol{f}^{\,\mathrm{pm}})$ for the overall equilibrium model proposed in Section 2.10, which is a block diagonal with $2 \times |\mathcal{A}|$ diagonal blocks as follows,

$$J\Phi'(\boldsymbol{f}^{\,\mathrm{am}}, \boldsymbol{f}^{\,\mathrm{pm}}) = \begin{bmatrix} J\Phi^{\,\prime;\,\mathrm{am}}(\boldsymbol{f}^{\,\mathrm{am}}) & \mathbf{0} \\ \\ \mathbf{0} & J\Phi^{\,\prime;\,\mathrm{pm}}(\boldsymbol{f}^{\,\mathrm{pm}}) \end{bmatrix}$$

where $J\Phi^{\,\prime;\,\mathrm{am}}(\boldsymbol{f}^{\,\mathrm{am}})$ is the Jacobian matrix for the morning arc flows, and $J\Phi^{\,\prime;\,\mathrm{pm}}(\boldsymbol{f}^{\,\mathrm{pm}})$ is the Jacobian matrix for the evening arc flows.

Note that the extended arc set is defined as $\mathcal{A} \triangleq \mathcal{A}_{\,\mathrm{sd}} \cup \mathcal{A}_{\,\mathrm{rd}} \cup \mathcal{A}_{\,\mathrm{rp}} \cup \mathcal{A}_{\,\mathrm{hp}}$ in the extended network in Fig. 2, which includes a "solo driver arc", a "rideshare driver arc", a "rideshare passenger arc" and a "ride-hailing passenger arc". Thus, both $J\Phi^{\,\prime;\,\mathrm{am}}(\boldsymbol{f}^{\,\mathrm{am}})$ and $J\Phi^{\,\prime;\,\mathrm{pm}}(\boldsymbol{f}^{\,\mathrm{pm}})$ consist of $4 \times |\mathcal{A}_0|$ diagonal blocks $\boldsymbol{B}^{\,\mathrm{am}}$. Take $J\Phi^{\,\prime;\,\mathrm{am}}(\boldsymbol{f}^{\,\mathrm{am}}) \triangleq \mathrm{diag}\,(\boldsymbol{B}^{\,\mathrm{am}}, \ldots, \boldsymbol{B}^{\,\mathrm{am}})$ as an example, each block $\boldsymbol{B}^{\,\mathrm{am}}$ is a Jacobian sub-matrix with respect to each arc $a_0 \in \mathcal{A}_0$, which can be written as

$$
\boldsymbol{B}^{\text{am}} = \begin{bmatrix}
\frac{\partial \text{tc}^{\text{am}}_{a_{\text{sd}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{sd}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{sd}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rd}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{sd}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rp}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{sd}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{hp}}}} \\
\frac{\partial \text{tc}^{\text{am}}_{a_{\text{rd}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{sd}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{rd}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rd}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{rd}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rp}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{rd}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{hp}}}} \\
\frac{\partial \text{tc}^{\text{am}}_{a_{\text{rp}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{sd}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{rp}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rd}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{rp}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rp}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{rp}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{hp}}}} \\
\frac{\partial \text{tc}^{\text{am}}_{a_{\text{hp}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{sd}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{hp}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rd}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{hp}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rp}}}} & \frac{\partial \text{tc}^{\text{am}}_{a_{\text{hp}}}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{hp}}}}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{sd}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rd}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rp}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{hp}}}} \\
\frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{sd}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rd}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rp}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{hp}}}} \\
\frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{sd}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rd}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rp}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{hp}}}} \\
\frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{sd}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rd}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{rp}}}} & \frac{\partial \text{tt}^{\text{am}}_{a_0}(\boldsymbol{f}^{\text{am}})}{\partial f^{\text{am}}_{a_{\text{hp}}}}
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{\partial \left(I^{\text{am}}_{a;\text{rd}}(\boldsymbol{f}^{\text{am}}) - R^{\text{am}}_{a;\text{rd}}(\boldsymbol{f}^{\text{am}})\right)}{\partial f^{\text{am}}_{a_{\text{rd}}}} & \frac{\partial \left(I^{\text{am}}_{a;\text{rd}}(\boldsymbol{f}^{\text{am}}) - R^{\text{am}}_{a;\text{rd}}(\boldsymbol{f}^{\text{am}})\right)}{\partial f^{\text{am}}_{a_{\text{rp}}}} & 0 \\
0 & \frac{\partial \left(I^{\text{am}}_{a;\text{rp}}(\boldsymbol{f}^{\text{am}}) + R^{\text{am}}_{a;\text{rp}}(\boldsymbol{f}^{\text{am}})\right)}{\partial f^{\text{am}}_{a_{\text{rd}}}} & \frac{\partial \left(I^{\text{am}}_{a;\text{rp}}(\boldsymbol{f}^{\text{am}}) + R^{\text{am}}_{a;\text{rp}}(\boldsymbol{f}^{\text{am}})\right)}{\partial f^{\text{am}}_{a_{\text{rp}}}} & 0 \\
0 & 0 & 0 & \frac{\partial \left(I^{\text{am}}_{a;\text{hp}}(\boldsymbol{f}^{\text{am}}) + R^{\text{am}}_{a;\text{hp}}(\boldsymbol{f}^{\text{am}})\right)}{\partial f^{\text{am}}_{a_{\text{hp}}}}
\end{bmatrix}
$$

where $a_{\text{t}} = \mathcal{T}_{\text{t}}(a_0)$ for $\text{t} \in \{\text{sd}, \text{rd}, \text{rp}, \text{hp}\}$.

For simplicity, denote $\frac{\partial}{\partial f^{\text{am}}_{a_{\text{t}}}}$ as ∂_{t}, then we have that

$$
\boldsymbol{B}^{\text{am}} = \begin{bmatrix}
\theta^{\text{am}}_a & \theta^{\text{am}}_a & 0 & \theta^{\text{am}}_a \\
\theta^{\text{am}}_a & \theta^{\text{am}}_a + \kappa^{\text{am}} \partial_{\text{rd}} S^{\text{am}}_{a;\text{rp}} & \partial_{\text{rp}} I^{\text{am}}_{a;\text{rd}} - \kappa^{\text{am}} \partial_{\text{rp}} E^{\text{am}}_{a;\text{rp}} & \theta^{\text{am}}_a \\
\theta^{\text{am}}_a & \theta^{\text{am}}_a - \partial_{\text{rd}} S^{\text{am}}_{a;\text{rp}} & \partial_{\text{rp}} I^{\text{am}}_{a;\text{rp}} + \partial_{\text{rp}} E^{\text{am}}_{a;\text{rp}} & \theta^{\text{am}}_a \\
\theta^{\text{am}}_a & \theta^{\text{am}}_a & 0 & \theta^{\text{am}}_a + \partial_{\text{hp}} I^{\text{am}}_{a;\text{hp}} + \partial_{\text{hp}} E^{\text{am}}_{a;\text{hp}}
\end{bmatrix}
$$

where $\theta_a^{\text{am}} = \frac{4t_a b}{\alpha_a{}^4}\left(f_{a_{\text{sd}}}^{\text{am}} + f_{a_{\text{rd}}}^{\text{am}} + f_{a_{\text{hp}}}^{\text{am}}\right)^3 \geq 0 \quad \forall a \in \mathcal{A}_0$ with $a_{\text{sd}} = \mathcal{T}_{\text{sd}}(a)$, $a_{\text{rd}} = \mathcal{T}_{\text{rd}}(a)$, $a_{\text{hp}} = \mathcal{T}_{\text{hp}}(a)$.

The matrix $\boldsymbol{B}^{\text{am}}$ is positive (semi)definite if its symmetric part of $\bar{\boldsymbol{B}}^{\text{am}} = \frac{1}{2}\left(\boldsymbol{B}^{\text{am}} + (\boldsymbol{B}^{\text{am}})^T\right)$ is positive (semi)definite. $\bar{\boldsymbol{B}}^{\text{am}}$ can be calculated as follows:

$$\bar{\boldsymbol{B}}^{\text{am}} = \begin{bmatrix} \theta_a^{\text{am}} & \theta_a^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} & \theta_a^{\text{am}} \\ \theta_a^{\text{am}} & \theta_a^{\text{am}} + \kappa^{\text{am}}\partial_{\text{rd}}S_{a;\text{rp}}^{\text{am}} & \frac{\theta_a^{\text{am}} - \partial_{\text{rd}}S_{a;\text{rp}}^{\text{am}} + \partial_{\text{rp}}I_{a;\text{rd}}^{\text{am}} - \kappa^{\text{am}}\partial_{\text{rp}}E_{a;\text{rp}}^{\text{am}}}{2} & \theta_a^{\text{am}} \\ \frac{\theta_a^{\text{am}}}{2} & \frac{\theta_a^{\text{am}} - \partial_{\text{rd}}S_{a;\text{rp}}^{\text{am}} + \partial_{\text{rp}}I_{a;\text{rd}}^{\text{am}} - \kappa^{\text{am}}\partial_{\text{rp}}E_{a;\text{rp}}^{\text{am}}}{2} & \partial_{\text{rp}}I_{a;\text{rp}}^{\text{am}} + \partial_{\text{rp}}E_{a;\text{rp}}^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} \\ \theta_a^{\text{am}} & \theta_a^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} & \theta_a^{\text{am}} + \partial_{\text{hp}}I_{a;\text{hp}}^{\text{am}} + \partial_{\text{hp}}E_{a;\text{hp}}^{\text{am}} \end{bmatrix}$$

Let $H_1^{\text{am}} \triangleq \kappa^{\text{am}}\partial_{\text{rd}}S_{a;\text{rp}}^{\text{am}}$, $H_2^{\text{am}} \triangleq \frac{-\partial_{\text{rd}}S_{a;\text{rp}}^{\text{am}} + \partial_{\text{rp}}I_{a;\text{rd}}^{\text{am}} - \kappa^{\text{am}}\partial_{\text{rp}}E_{a;\text{rp}}^{\text{am}}}{2}$, $H_3^{\text{am}} \triangleq \partial_{\text{rp}}I_{a;\text{rp}}^{\text{am}} + \partial_{\text{rp}}E_{a;\text{rp}}^{\text{am}}$, $H_4^{\text{am}} \triangleq \partial_{\text{hp}}I_{a;\text{hp}}^{\text{am}} + \partial_{\text{hp}}E_{a;\text{hp}}^{\text{am}}$. Then the matrix $\bar{\boldsymbol{B}}^{\text{am}}$ can be written as follows:

$$\bar{\boldsymbol{B}}^{\text{am}} = \begin{bmatrix} \theta_a^{\text{am}} & \theta_a^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} & \theta_a^{\text{am}} \\ \theta_a^{\text{am}} & \theta_a^{\text{am}} + H_1^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} + H_2^{\text{am}} & \theta_a^{\text{am}} \\ \frac{\theta_a^{\text{am}}}{2} & \frac{\theta_a^{\text{am}}}{2} + H_2^{\text{am}} & H_3^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} \\ \theta_a^{\text{am}} & \theta_a^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} & \theta_a^{\text{am}} + H_4^{\text{am}} \end{bmatrix}$$

Similarly, we can derive such matrix $\bar{\boldsymbol{B}}^{\text{pm}}$ for $J\Phi'^{;\text{pm}}(\boldsymbol{f}^{\text{pm}})$ consisting of θ_a^{pm}, H_1^{pm}, H_2^{pm}, H_3^{pm}, H_4^{pm}.

In Proposition 1 below, we provide the conditions under which the matrix $\bar{\boldsymbol{B}}^{\text{am}}$ and $\bar{\boldsymbol{B}}^{\text{pm}}$ will be positive definite, and as a result, our proposed model will have a unique solution. When $\theta_a^{\text{am}} \neq 0$ and $\theta_a^{\text{pm}} \neq 0$ for all $a \in \mathcal{A}_0$, namely all arcs in the network are used by the travelers (note that this is a common assumption in existing literature such as Section 3.3 of Sheffi, 1985; Xu et al., 2015b; Ma et al., 2020; Li et al., 2020; Wang et al., 2021), we provide the conditions under which the model will have a globally unique solution, see Proposition 1(i). For the situation $\theta_a^{\text{am}} = 0$ or $\theta_a^{\text{pm}} = 0$ for some $a \in \mathcal{A}_0$, namely some arcs are not used by travelers, global uniqueness will no longer be possible. Instead, we show that under the same condition, the local unique solution can be achieved, see Proposition 1(ii).

Proposition 1. *Under the following condition:*

$$\frac{4t_a b}{\alpha_a} \times \left(\frac{\sum_{k\in\mathcal{K}} D_k}{\alpha_a}\right)^3 < \frac{4H_1^{\text{am}} H_3^{\text{am}} - 4(H_2^{\text{am}})^2}{H_1^{\text{am}}} \quad \forall a \in \mathcal{A}_0$$

$$\frac{4t_a b}{\alpha_a} \times \left(\frac{\sum_{k\in\mathcal{K}} D_k}{\alpha_a}\right)^3 < \frac{4H_1^{\text{pm}} H_3^{\text{pm}} - 4(H_2^{\text{pm}})^2}{H_1^{\text{pm}}} \quad \forall a \in \mathcal{A}_0$$

where $a_{\text{sd}} = \mathcal{T}_{\text{sd}}(a)$, $a_{\text{rd}} = \mathcal{T}_{\text{rd}}(a)$, $a_{\text{hp}} = \mathcal{T}_{\text{hp}}(a)$.

(i) If $\theta_a^{\text{am}} = \frac{4t_a b}{\alpha_a} \times \left(\frac{f_{a_{\text{sd}}}^{\text{am}}+f_{a_{\text{rd}}}^{\text{am}}+f_{a_{\text{hp}}}^{\text{am}}}{\alpha_a}\right)^3 \neq 0$ *and* $\theta_a^{\text{pm}} = \frac{4t_a b}{\alpha_a} \times \left(\frac{f_{a_{\text{sd}}}^{\text{pm}}+f_{a_{\text{rd}}}^{\text{pm}}+f_{a_{\text{hp}}}^{\text{pm}}}{\alpha_a}\right)^3 \neq 0$ *for all* $a \in \mathcal{A}_0$, *we will have globally unique arc flows, namely* $f_{a_{\text{sd}}}^{\text{am/pm}}$, $f_{a_{\text{rd}}}^{\text{am/pm}}$, $f_{a_{\text{rp}}}^{\text{am/pm}}$, *and* $f_{a_{\text{hp}}}^{\text{am/pm}}$ *are globally unique.*

(ii) If $\theta_a^{\text{am}} = \frac{4t_a b}{\alpha_a} \times \left(\frac{f_{a_{\text{sd}}}^{\text{am}}+f_{a_{\text{rd}}}^{\text{am}}+f_{a_{\text{hp}}}^{\text{am}}}{\alpha_a}\right)^3 = 0$ *or* $\theta_a^{\text{pm}} = \frac{4t_a b}{\alpha_a} \times \left(\frac{f_{a_{\text{sd}}}^{\text{pm}}+f_{a_{\text{rd}}}^{\text{pm}}+f_{a_{\text{hp}}}^{\text{pm}}}{\alpha_a}\right)^3 = 0$ *for some* $a \in \mathcal{A}_0$, *we will have locally unique arc flows, namely* $f_{a_{\text{sd}}}^{\text{am/pm}}$, $f_{a_{\text{rd}}}^{\text{am/pm}}$, $f_{a_{\text{rp}}}^{\text{am/pm}}$, *and* $f_{a_{\text{hp}}}^{\text{am/pm}}$ *are locally unique.*

Proof. Here we provide the proof for matrix $\bar{\boldsymbol{B}}^{\text{am}}$, and the proof for matrix $\bar{\boldsymbol{B}}^{\text{pm}}$ is similar.

(i) Based on Theorem 2.3.3(a) of Facchinei and Pang (2003), we need the matrix $\bar{\boldsymbol{B}}^{\text{am}}$ to be positive definite to derive globally unique arc flows, which is equivalent to show that all its upper left submatrices have positive determinants. The upper left 1×1 and 2×2 determinants of the matrix $\bar{\boldsymbol{B}}^{\text{am}}$ are positive when $\theta_a^{\text{am}} > 0$. From that upper left 3×3 determinant of the matrix $\bar{\boldsymbol{B}}^{\text{am}}$ is positive we have

$$\begin{vmatrix} \theta_a^{\text{am}} & \theta_a^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} \\ \theta_a^{\text{am}} & \theta_a^{\text{am}} + H_1^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} + H_2^{\text{am}} \\ \frac{\theta_a^{\text{am}}}{2} & \frac{\theta_a^{\text{am}}}{2} + H_2^{\text{am}} & H_3^{\text{am}} \end{vmatrix} = -\frac{1}{4}\theta_a^{\text{am}} \left(H_1^{\text{am}}\theta_a^{\text{am}} - 4H_1^{\text{am}}H_3^{\text{am}} + 4(H_2^{\text{am}})^2 \right) > 0$$

$$\Rightarrow \theta_a^{\text{am}} < \frac{4H_1^{\text{am}}H_3^{\text{am}} - 4(H_2^{\text{am}})^2}{H_1^{\text{am}}}$$

Since the upper left 4×4 determinant of the matrix $\bar{\boldsymbol{B}}^{\text{am}}$ is positive we have

$$\begin{vmatrix} \theta_a^{\text{am}} & \theta_a^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} & \theta_a^{\text{am}} \\ \theta_a^{\text{am}} & \theta_a^{\text{am}} + H_1^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} + H_2^{\text{am}} & \theta_a^{\text{am}} \\ \frac{\theta_a^{\text{am}}}{2} & \frac{\theta_a^{\text{am}}}{2} + H_2^{\text{am}} & H_3^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} \\ \theta_a^{\text{am}} & \theta_a^{\text{am}} & \frac{\theta_a^{\text{am}}}{2} & \theta_a^{\text{am}} + H_4^{\text{am}} \end{vmatrix}$$

$$= -\frac{1}{4} \times H_4^{\text{am}} \times \theta_a^{\text{am}} \left(H_1^{\text{am}} \theta_a^{\text{am}} - 4 H_1^{\text{am}} H_3^{\text{am}} + 4 (H_2^{\text{am}})^2 \right) > 0$$

$$\Rightarrow \theta_a^{\text{am}} < \frac{4 H_1^{\text{am}} H_3^{\text{am}} - 4 (H_2^{\text{am}})^2}{H_1^{\text{am}}}$$

Thus, the condition for matrix $\bar{\boldsymbol{B}}^{\text{am}}$ to be positive definite is

$$\theta_a^{\text{am}} = \frac{4 t_a b}{\alpha_a} \times \left(\frac{f_{a_{\text{sd}}}^{\text{am}} + f_{a_{\text{rd}}}^{\text{am}} + f_{a_{\text{hp}}}^{\text{am}}}{\alpha_a} \right)^3 < \frac{4 H_1^{\text{am}} H_3^{\text{am}} - 4 (H_2^{\text{am}})^2}{H_1^{\text{am}}}$$

To satisfy the condition above, from $\frac{f_{a_{\text{sd}}}^{\text{am}} + f_{a_{\text{rd}}}^{\text{am}} + f_{a_{\text{hp}}}^{\text{am}}}{\alpha_a} \leq \frac{\sum_{k \in \mathcal{K}} D_k}{\alpha_a}$ for all $a \in \mathcal{A}_0$ we need to have that

$$\frac{4 t_a b}{\alpha_a} \times \left(\frac{\sum_{k \in \mathcal{K}} D_k}{\alpha_a} \right)^3 < \frac{4 H_1^{\text{am}} H_3^{\text{am}} - 4 (H_2^{\text{am}})^2}{H_1^{\text{am}}} \quad \forall a \in \mathcal{A}_0$$

(ii) When $\theta_a^{\text{am}} = 0$ for some a. With $\theta_a^{\text{am}} = 0$, the matrix $\bar{\boldsymbol{B}}^{\text{am}}$ can be written as follows:

$$\bar{\boldsymbol{B}}^{\text{am}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & H_1^{\text{am}} & H_2^{\text{am}} & 0 \\ 0 & H_2^{\text{am}} & H_3^{\text{am}} & 0 \\ 0 & 0 & 0 & H_4^{\text{am}} \end{bmatrix}$$

Since $\det(\bar{\boldsymbol{B}}^{\text{am}}) = 0$, under the conditions that $\frac{4 t_a b}{\alpha_a} \times \left(\frac{\sum_{k \in \mathcal{K}} D_k}{\alpha_a} \right)^3 < \frac{4 H_1^{\text{am}} H_3^{\text{am}} - 4 (H_2^{\text{am}})^2}{H_1^{\text{am}}} \quad \forall a \in \mathcal{A}_0$, from the proof of (i) we know that the matrix $\bar{\boldsymbol{B}}^{\text{am}}$ is positive semidefinite but not positive definite. In this situation, there is no hope for global uniqueness of arc flows. Instead, we try to achieve the second best property, which is to derive local uniqueness of arc flows.

Let $H_2' \triangleq \partial_{\text{rp}} I_{a;\text{rd}}^{\text{am}} - \kappa^{\text{am}} \partial_{\text{rp}} E_{a;\text{rp}}^{\text{am}}$ and $H_2'' \triangleq -\partial_{\text{rd}} S_{a;\text{rp}}^{\text{am}}$. When $\theta_a^{\text{am}} = 0$, the matrix $\boldsymbol{B}^{\text{am}}$ can be written as follows:

$$\boldsymbol{B}^{\text{am}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & H_1 & H_2' & 0 \\ 0 & H_2'' & H_3 & 0 \\ 0 & 0 & 0 & H_4 \end{bmatrix}$$

When $\theta_a^{\text{am}} = 0$, we must have that $f_{a_{\text{sd}}}^{\text{am}} = f_{a_{\text{rd}}}^{\text{am}} = f_{a_{\text{rp}}}^{\text{am}} = f_{a_{\text{hp}}}^{\text{am}} = 0$. Let $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$.

Since the arc cost function (8) is convex, it is locally lipschitz at the point $\mathbf{0}$. Since the arc cost function (8) is also directional derivative at the point $\mathbf{0}$, by definition 3.1.2 of Facchinei and Pang (2003), it is B-differentiable at the point $\mathbf{0}$. Thus, Proposition 3.3.7 of Facchinei and Pang (2003) applies.

According to Proposition 3.3.7(a) of Facchinei and Pang (2003), to show a given solution $\boldsymbol{f}^{\text{am}}$ is locally unique, we need to show that the following homogeneous Complementarity Problem (CP) has $\boldsymbol{f}^{\text{am}} = 0$ as the unique solution:

$$\mathcal{C}(\mathbf{0}, \mathcal{HF}', \Phi') \ni \boldsymbol{f}^{\text{am}} \perp J\Phi'^{;\text{am}}(\mathbf{0})\boldsymbol{f}^{\text{am}} \in \mathcal{C}(\mathbf{0}, \mathcal{HF}', \Phi')^*$$

where $\mathcal{C}(\mathbf{0}, \mathcal{HF}', \Phi')$ is a critical cone and $\mathcal{C}(\mathbf{0}, \mathcal{HF}', \Phi')^*$ represents its dual cone.

From $(\boldsymbol{f}^{\text{am}})^T J\Phi'^{;\text{am}}(\mathbf{0})\boldsymbol{f}^{\text{am}} = 0$ we have that

$$(\boldsymbol{f}^{\text{am}})^T \boldsymbol{B}^{\text{am}} \boldsymbol{f}^{\text{am}} = 0$$

$$\Rightarrow \begin{bmatrix} f_{a_{\text{rd}}}^{\text{am}} & f_{a_{\text{rp}}}^{\text{am}} & f_{a_{\text{hp}}}^{\text{am}} \end{bmatrix} \begin{bmatrix} H_1 & H_2' & 0 \\ H_2'' & H_3 & 0 \\ 0 & 0 & H_4 \end{bmatrix} \begin{bmatrix} f_{a_{\text{rd}}}^{\text{am}} \\ f_{a_{\text{rp}}}^{\text{am}} \\ f_{a_{\text{hp}}}^{\text{am}} \end{bmatrix} = 0$$

Under the conditions of Proposition 1, the 3×3 matrix in the middle above is positive definite. Thus we have that

$$f_{a_{\text{rd}}}^{\text{am}} = f_{a_{\text{rp}}}^{\text{am}} = f_{a_{\text{hp}}}^{\text{am}} = 0$$

From $\boldsymbol{f}^{\text{am}} \in \mathcal{C}(\mathbf{0}, \mathcal{HF}', \Phi')$, by definition we have

$$f_{a_{\text{sd}}}^{\text{am}} \, \text{tt}_{a_{\text{sd}}}^{\text{am}}(\mathbf{0}) + f_{a_{\text{rd}}}^{\text{am}} \, \text{tt}_{a_{\text{rd}}}^{\text{am}}(\mathbf{0}) + f_{a_{\text{rp}}}^{\text{am}} \, \text{tt}_{a_{\text{rp}}}^{\text{am}}(\mathbf{0}) + f_{a_{\text{hp}}}^{\text{am}} \, \text{tt}_{a_{\text{hp}}}^{\text{am}}(\mathbf{0}) = 0$$

Since $f_{a_{\text{rd}}}^{\text{am}} = f_{a_{\text{rp}}}^{\text{am}} = f_{a_{\text{hp}}}^{\text{am}} = 0$ we must have

$$f_{a_{\text{sd}}}^{\text{am}} \, \text{tt}_{a_{\text{sd}}}^{\text{am}}(\mathbf{0}) = 0$$

$$\Rightarrow f_{a_{\text{sd}}}^{\text{am}} = 0$$

Thus, $\boldsymbol{f}^{\text{am}} = 0$ is the unique solution for the homogeneous CP above-mentioned. And we have locally unique arc flows when $\theta_a^{\text{am}} = 0$ for some $a \in \mathcal{A}_0$. □

4 Comparison with A Decoupled Modeling Approach

In the previous section, we have developed a model for coupled morning-evening commute, which we call the **coupled model**. In this section, we compare the equilibrium solution from the coupled model with a decoupled modeling approach.

Let the **decoupled morning model** be one that solves both the route and mode choice for the am given a set of OD demands and the **decoupled evening model** be one that solves both the route and mode choice for the pm given a set of OD demands. Identifying an equilibrium solution to these two problems separately and then combining them together would most likely violate the constraint that a traveler if chooses to be a driver he/she must be a driver both in the am and pm. Thus, the two decoupled models must be linked in some manner.

Before presenting our approach for linking the models, we present two other models. Let the **constrained decoupled morning model** be one that solves the route choice and partial mode choice (e.g., rideshare and ride-hailing passengers) for the am given a set of demands and set of drivers for each OD pair and the **constrained decoupled evening model** be one that solves the route choice and partial mode choice (e.g., rideshare and ride-hailing passengers) for the pm given a set of demands and drivers for each OD pair.

For a decoupled approach, we assume a traveler uses one of the decoupled models to determine whether they become a driver or not. Based on this assumption, we present two decoupled solution approaches.

Solution D1

Step 1: Identify an equilibrium solution to the **decoupled morning model** to determine for each OD pair the set of routes and mode choice (solo drivers, rideshare drivers, rideshare passengers, and ride-hailing passengers) for the am only. Mathematically, the **decoupled morning model** includes the morning part of Equations (10), (14), (15), (16), and (19).

Step 2: Given the total number of drivers for each OD pair from the solution in Step 1, identify an equilibrium solution to the **constrained evening model** to determine for each OD the set of routes and mode choice (in particular, rideshare passengers and ride-hailing passengers) for the pm only. Mathematically, the **constrained evening model** includes the evening part of Equations (10), (14), (15), (16), (19), and the driver flow conservation constraint (17).

Solution D2

Step 1: Identify an equilibrium solution to the **decoupled evening model** to determine for each OD pair the set of routes and mode choice (solo drivers, rideshare drivers, rideshare passengers, and ride-hailing passengers) for the pm only. Mathematically, the **decoupled evening model** includes the evening part of Equations (10), (14), (15), (16), and (19).

Step 2: Given the total number set of drivers for each OD pair from the solution in Step 1, identify an equilibrium solution to the **constrained morning model** to determine for each OD the set of routes and mode choice (in particular, rideshare passengers and ride-hailing passengers) for the am only. Mathematically, the **constrained morning model** includes the morning part of Equations (10), (14), (15), (16), (19), and the driver flow conservation constraint (17).

We next prove that a traveler's disutility will not be greater if they follow the equilibrium solution of the **coupled model** over solution D1 or solution D2. We prove the conclusion for solution D2, and the proof for solution D1 is similar.

Proposition 2. *Under the condition of Proposition 1(i), a traveler's disutility derived from the coupled model will not be greater than that of the decoupled model based on solution D2, namely,*

$$u_k \leq \hat{u}_k^{\text{am}} + \hat{u}_{\bar{k}}^{\text{pm}} \quad \forall\, k \in \mathcal{K} \textit{ and associated } \bar{k}$$

where u_k is the least disutility of the coupled model, $\hat{u}_{\bar{k}}^{\text{pm}}$ is the least disutility of the decoupled evening *model, and* $\hat{u}_k^{\text{am}}$ *is the least disutility of the constrained morning model. Since the costs are additive, the least disutility for each OD pair (*u_k*,* $\hat{u}_{\bar{k}}^{\text{pm}}$*,* $\hat{u}_k^{\text{am}}$*) is the summation of the least disutility of all the arcs consisting the path joining that OD pair.*

Proof. Under the condition of Proposition 1(i), the arc cost functions $\text{tc}_a^{\text{am}}(\boldsymbol{f}^{\,\text{am}})$ and $\text{tc}_a^{\text{pm}}(\boldsymbol{f}^{\,\text{pm}})$ are strictly monotone. Thus, according to Definition 2.3.9 (d) of Facchinei and Pang (2003), the path cost function $\text{TC}_p(\boldsymbol{z}) = (\boldsymbol{\Delta}^{\text{am}})^T \text{tc}_a^{\text{am}}(\boldsymbol{\Delta}^{\text{am}}\boldsymbol{z}) + (\boldsymbol{\Delta}^{\text{pm}})^T \text{tc}_a^{\text{pm}}(\boldsymbol{\Delta}^{\text{pm}}\boldsymbol{z})$ is strictly monotone composite. Since strictly monotone composite gives us pseudo monotone plus, then the path cost function $\text{TC}_p(\boldsymbol{z})$ is also pseudo monotone plus. According to Corollary 2.3.10 of Facchinei and Pang (2003), we can conclude that the solution set of the coupled model is F-unique, namely $\text{TC}_p(\boldsymbol{z})$ is a singleton.

Under the condition of Proposition 1(i), the arc flows $\boldsymbol{f}^{\,\text{am}}$ and $\boldsymbol{f}^{\,\text{pm}}$ are globally unique. As a result, the shadow prices derived from Equation (12) are also unique. Then from the user equilibrium conditions (18) of the coupled model, we know that u_k must be unique for each OD pair $k \in \mathcal{K}$.

Let the feasible solution set of the coupled model be Ξ. Then the coupled model is equivalent to the multi-agent optimization problem as follows (otherwise the F-uniqueness property will be violated):

$$\min u_k \quad \text{s.t.} \quad u_k \in \Xi \quad \forall\, k \in \mathcal{K}$$

where each OD pair is an agent. Here each agent wants to minimize its least disutility, which is consistent with the user equilibrium conditions of the coupled model.

Let the feasible solution set of the **decoupled evening model** and the **constrained morning model** be Ξ^{pm} and Ξ^{am}, respectively. We have the following:

$$\Xi = \Xi^{\text{pm}} \cup \Xi^{\text{am}}$$

Consider the two multi-agent optimization problems below,

$$\min \hat{u}_{\bar{k}}^{\text{pm}} \quad \text{s.t.} \quad \hat{u}_{\bar{k}}^{\text{pm}} \in \Xi^{\text{pm}} \quad \forall \text{ associated } \bar{k}$$

$$\min \hat{u}_k^{\text{am}} \quad \text{s.t.} \quad \hat{u}_k^{\text{am}} \in \Xi^{\text{am}} \quad \forall\, k \in \mathcal{K}$$

where the driver flow conservation constraint (17) is included in Ξ^{am}.

For each $k \in \mathcal{K}$ and associated $\bar{k}$, the two multi-agent optimization problems above can be viewed as two sequentially optimized subproblems for the equivalent multi-agent optimization problem of the coupled model.

Since two sequentially optimized subproblems cannot lead to better objective value compared with the original problem, we have that

$$u_k = \min_{u_k \in \Xi} u_k \leq \min_{\hat{u}_k^{\text{am}} \in \Xi^{\text{am}}} \hat{u}_k^{\text{am}} + \min_{\hat{u}_{\bar{k}}^{\text{pm}} \in \Xi^{\text{pm}}} \hat{u}_{\bar{k}}^{\text{pm}} \leq \hat{u}_k^{\text{am}} + \hat{u}_{\bar{k}}^{\text{pm}} \quad \forall\, k \in \mathcal{K} \text{ and associated } \bar{k}$$

which means that a traveler's disutility derived from the coupled model will not be greater than that of the decoupled model. □

5 Computational Results

In this section, we use the well-known Sioux-Falls network to test the proposed model. The numerical experiments are derived by solving the equivalent MiCP formulation of the arc-based VI model in Section 2.9 (see Appendix 1). The results in this section are obtained by solving the MiCP using Knitro (Byrd et al. 2006) on the NEOS server.

For the experiments, we use functions for inconvenience costs and payments similar to those defined by Xu et al. (2015b) as follows:

- *Inconvenience cost of rideshare drivers*:

$$\left.\begin{aligned} I_{a;\text{rd}}^{\text{am}}\left(\boldsymbol{f}^{\text{am}}\right) &\triangleq \gamma_{\text{rd}}^{\text{am}} f_{a_{\text{rp}}}^{\text{am}} \\ I_{a;\text{rd}}^{\text{pm}}\left(\boldsymbol{f}^{\text{pm}}\right) &\triangleq \gamma_{\text{rd}}^{\text{pm}} f_{a_{\text{rp}}}^{\text{pm}} \end{aligned}\right\} \quad \forall\, a \in \mathcal{A}_{\text{rd}} \text{ with } a_{\text{rp}} = \mathcal{T}_{\text{rp}}\left(\mathcal{T}_0\left(a\right)\right) \tag{20}$$

where the constants $\gamma_{\text{rd}}^{\text{am}}$ and $\gamma_{\text{rd}}^{\text{pm}}$ are positive.

- *Inconvenience cost of rideshare passengers*:

$$\left.\begin{aligned} I_{a;\text{rp}}^{\text{am}}\left(\boldsymbol{f}^{\text{am}}\right) &\triangleq \gamma_{\text{rp}}^{\text{am}} f_a \\ I_{a;\text{rp}}^{\text{pm}}\left(\boldsymbol{f}^{\text{pm}}\right) &\triangleq \gamma_{\text{rp}}^{\text{pm}} f_a \end{aligned}\right\} \quad \forall\, a \in \mathcal{A}_{\text{rp}} \tag{21}$$

where the constants $\gamma_{\text{rp}}^{\text{am}}$ and $\gamma_{\text{rp}}^{\text{pm}}$ are positive.

- *Inconvenience cost of ride-hailing passengers*:

$$\left.\begin{aligned} I_{a;\mathrm{hp}}^{\mathrm{am}}\left(\boldsymbol{f}^{\mathrm{am}}\right) &\triangleq \gamma_{\mathrm{hp}}^{\mathrm{am}}\, f_a^{\mathrm{am}} \\ I_{a;\mathrm{hp}}^{\mathrm{pm}}\left(\boldsymbol{f}^{\mathrm{pm}}\right) &\triangleq \gamma_{\mathrm{hp}}^{\mathrm{pm}}\, f_a^{\mathrm{pm}} \end{aligned}\right\} \quad \forall\, a \in \mathcal{A}_{\mathrm{hp}} \tag{22}$$

where the constants $\gamma_{\mathrm{hp}}^{\mathrm{am}}$ and $\gamma_{\mathrm{hp}}^{\mathrm{pm}}$ are positive.

- *Payment of rideshare passengers*:

$$\left.\begin{aligned} R_{a;\mathrm{rp}}^{\mathrm{am}}\left(\boldsymbol{f}^{\mathrm{am}}\right) &\triangleq \rho_{\mathrm{rp}}^{\mathrm{am}}\, t_{a_0} - v_{\mathrm{rp}}^{\mathrm{am}}\, f_{a_{\mathrm{rd}}}^{\mathrm{am}} + w_{\mathrm{rp}}^{\mathrm{am}}\, f_a^{\mathrm{am}} \\ R_{a;\mathrm{rp}}^{\mathrm{pm}}\left(\boldsymbol{f}^{\mathrm{pm}}\right) &\triangleq \rho_{\mathrm{rp}}^{\mathrm{pm}}\, t_{a_0} - v_{\mathrm{rp}}^{\mathrm{pm}}\, f_{a_{\mathrm{rd}}}^{\mathrm{pm}} + w_{\mathrm{rp}}^{\mathrm{pm}}\, f_a^{\mathrm{pm}} \end{aligned}\right\} \quad \forall\, a \in \mathcal{A}_{\mathrm{rp}} \tag{23}$$

where $a_{\mathrm{rd}} = \mathcal{T}_{\mathrm{rd}}(\mathcal{T}_0(a))$ for $a \in \mathcal{A}_{\mathrm{rp}}$ is the corresponding rideshare driver arc corresponding to the rideshare passenger arc a, $a_0 = \mathcal{T}_0(a)$ is the corresponding original arc to the rideshare-passenger arc $a \in \mathcal{A}_{\mathrm{rp}}$. The constants $\rho_{\mathrm{rp}}^{\mathrm{am}}$, $\rho_{\mathrm{rp}}^{\mathrm{pm}}$, $v_{\mathrm{rp}}^{\mathrm{am}}$, $v_{\mathrm{rp}}^{\mathrm{pm}}$, $w_{\mathrm{rp}}^{\mathrm{am}}$, and $w_{\mathrm{rp}}^{\mathrm{pm}}$ are positive. The parameters t_a are the free flow travel time of arc $a \in \mathcal{A}_0$.

- *Income of rideshare drivers*:

$$\left.\begin{aligned} R_{a;\mathrm{rd}}^{\mathrm{am}}\left(\boldsymbol{f}^{\mathrm{am}}\right) &\triangleq \kappa^{\mathrm{am}}\, R_{a;\mathrm{rp}}^{\mathrm{am}}\left(\boldsymbol{f}^{\mathrm{am}}\right) \\ R_{a;\mathrm{rd}}^{\mathrm{pm}}\left(\boldsymbol{f}^{\mathrm{pm}}\right) &\triangleq \kappa^{\mathrm{pm}}\, R_{a;\mathrm{rp}}^{\mathrm{pm}}\left(\boldsymbol{f}^{\mathrm{pm}}\right) \end{aligned}\right\} \quad \forall\, a \in \mathcal{A}_{\mathrm{rd}} \tag{24}$$

where the constants $\kappa^{\mathrm{am}}, \kappa^{\mathrm{pm}} \in [1, M]$.

- *Payment of ride-hailing passengers*:

$$\left.\begin{aligned} R_{a;\mathrm{hp}}^{\mathrm{am}}\left(\boldsymbol{f}^{\mathrm{am}}\right) &\triangleq \rho_{\mathrm{hp}}^{\mathrm{am}}\, t_{a_0} + w_{\mathrm{hp}}^{\mathrm{am}}\, f_a^{\mathrm{am}} \\ R_{a;\mathrm{hp}}^{\mathrm{pm}}\left(\boldsymbol{f}^{\mathrm{pm}}\right) &\triangleq \rho_{\mathrm{hp}}^{\mathrm{pm}}\, t_{a_0} + w_{\mathrm{hp}}^{\mathrm{pm}}\, f_a^{\mathrm{pm}} \end{aligned}\right\} \quad \forall\, a \in \mathcal{A}_{\mathrm{hp}} \tag{25}$$

where $a_0 = \mathcal{T}_0(a)$ is the corresponding original arc to the ridesourcing-passenger arc $a \in \mathcal{A}_{\mathrm{hp}}$. The constants $\rho_{\mathrm{hp}}^{\mathrm{am}}$, $\rho_{\mathrm{hp}}^{\mathrm{pm}}$, $w_{\mathrm{hp}}^{\mathrm{am}}$, and $w_{\mathrm{hp}}^{\mathrm{pm}}$ are positive. The parameters t_a are the free flow travel time of arc $a \in \mathcal{A}_0$.

Model parameters in terms of travel modes are set based on the following guidelines: (1) parameters for inconvenience of rideshare passengers are no smaller than those of ride-hailing passengers, e.g., $\gamma_{\mathrm{rp}}^{\mathrm{am}} \geq \gamma_{\mathrm{hp}}^{\mathrm{am}}$; (2) parameters for payment of rideshare passengers are no larger than those of ride-hailing passengers, e.g., $w_{\text{rp}}^{\text{am}} \leq w_{\text{hp}}^{\text{am}}$. In addition, we set the parameters to satisfy the conditions in Proposition 1 in order to guarantee solution uniqueness.

For the settings of the Sioux-Falls network, we follow Ben (2020), including the geometry, travel demand for each OD pair, and parameters of the BPR function for each arc. The original network has 76 arcs and 24 nodes. After constructing the extended network, there will be $76 \times 4 = 304$ arcs and $24 \times 3 = 72$ nodes. We randomly generate 200 out of the 528 OD pairs for the analysis, which is larger than the size tested in Xu et al. (2015b), Ban et al. (2019), Di and Ban (2019) and Noruzoliaee and Zou (2022). Model parameters for the base case are listed in Table 4, which is from Xu et al. (2015b). The conversion factor of time to money, ψ, is set to be 30 dollars/hour. The computation time for solving each case ranges from one to three hours.

Table 4. Parameters of the base case.

Parameters	Value	Units
$\gamma_{\text{rd}}^{\text{am}}$, $\gamma_{\text{rd}}^{\text{pm}}$	0.01	Dollars
$\gamma_{\text{rp}}^{\text{am}}$, $\gamma_{\text{rp}}^{\text{pm}}$	0.01	Dollars
$\gamma_{\text{hp}}^{\text{am}}$, $\gamma_{\text{hp}}^{\text{pm}}$	0.001	Dollars
κ^{am}, κ^{pm}	2	Persons
$\rho_{\text{rp}}^{\text{am}}$, $\rho_{\text{rp}}^{\text{pm}}$	0.5	Dollars
$v_{\text{rp}}^{\text{am}}$, $v_{\text{rp}}^{\text{pm}}$	0.2	Dollars
$w_{\text{rp}}^{\text{am}}$, $w_{\text{rp}}^{\text{pm}}$	0.1	Dollars
$\rho_{\text{hp}}^{\text{am}}$, $\rho_{\text{hp}}^{\text{pm}}$	0.5	Dollars
$w_{\text{hp}}^{\text{am}}$, $w_{\text{hp}}^{\text{pm}}$	0.15	Dollars

Table 5 and Table 6 are derived from the base case, in which the morning and evening parameters are the same. Table 5 shows the travelers' mode choice and mode switches in the morning commute and evening commute for the base case. We note that although parameters for the morning and evening are the same, the travelers' mode choice could be different. The reasons are that (i) the road networks are not symmetrical for the morning and evening trips; (ii) the Sioux-Falls network is not symmetrical. The proposed model shows that 7.0% of the rideshare drivers in the morning switch to solo drivers in the evening, which leads to a more expensive rideshare price in the evening. As a result, 6.8% of rideshare passengers in the morning switch to ride-hailing passengers in the evening.

Table 5. Travelers' mode choice and mode switches.

	AM	PM
# Solo Drivers	14905	16111
# Rideshare Drivers	17335	16129
# Rideshare Passengers	36968	34455
# Ride-hailing Passengers	13901	16414
# AM Rideshare Drivers ⇒ PM Solo Drivers	1206	
# AM Rideshare Passengers ⇒ PM Ride-hailing Passengers	2513	

Table 6 gives us the proportion of rideshare drivers that choose detour in the morning and evening commutes. We note that there are 9.0% more rideshare drivers that choose to take a detour in the morning because the rideshare payment for detour is higher in the morning, which compensates the rideshare drivers to take a detour.

Table 6. Proportion of rideshare drivers with detour.

	AM	PM
Rideshare Drivers with detour	30.9%	21.9%

Fig. 3 shows how the Vehicle Hours Traveled (VHT) change when we change the evening rideshare payment ($w_{\mathrm{rp}}^{\mathrm{pm}}$) and evening rideshare driver's inconvenience ($\gamma_{\mathrm{rp}}^{\mathrm{pm}}$), respectively. As we can see, the change of evening parameters influences not only the evening VHT but also the morning VHT because of the interactions between the morning and evening commutes. When the evening parameters change, the evening traffic flows change; since the evening traffic flows have impact on the morning, the morning traffic flows also change; and from an aggregate level, we observe that the morning VHT changes. In Fig. 3(b) we can see that even if we only change the evening rideshare driver's inconvenience, the morning VHT changes as rapidly as that of the evening. Without the coupled model, we can calculate the traffic equilibria for the morning and evening separately, which may underestimate the morning VHT when we change the evening rideshare payment ($w_{\mathrm{rp}}^{\mathrm{pm}}$) or evening rideshare driver's inconvenience ($\gamma_{\mathrm{rp}}^{\mathrm{pm}}$).

The coupling effects between morning and evening can also be observed in travelers' mode choice. The changes of VHTs in Fig. 3 can be explained by the changes of travelers' mode choice, as shown in Fig. 4 and Fig. 5, respectively. Fig. 4 shows us the sensitivity analysis for travelers' mode choice when changing the rideshare payment ($w_{\mathrm{rp}}^{\mathrm{pm}}$) in the evening. We can observe from Fig. 4(b) that as the evening rideshare payment increases, the number of rideshare passengers decreases. As a result, the rideshare market in the evening becomes smaller and we do not need so many rideshare drivers. More of the rideshare drivers and rideshare passengers switch to solo drivers than ride-hailing passengers in the evening, which leads to an increase in the total number of drivers in the evening. With the coupling effects between the morning and evening, the number of morning rideshare passengers decreases and the total number of morning drivers increases, as shown in Fig. 4(a). Here more travelers choose to drive in the morning because they know that it would be expensive to be a passenger in the evening. Since there are fewer rideshare drivers and passengers in the system, we can expect larger VHTs as in Fig. 3(a).

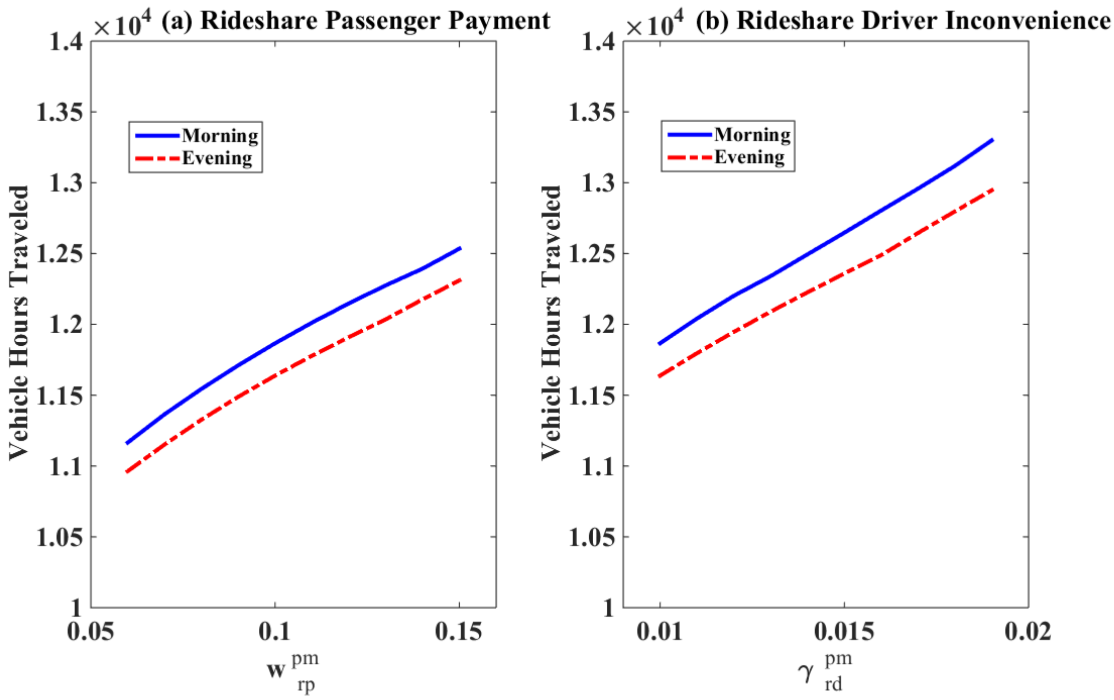

Figure 3. Results of Vehicle Hours Traveled when changing (a) $w_{\mathrm{rp}}^{\mathrm{pm}}$; (b) $\gamma_{\mathrm{rd}}^{\mathrm{pm}}$.

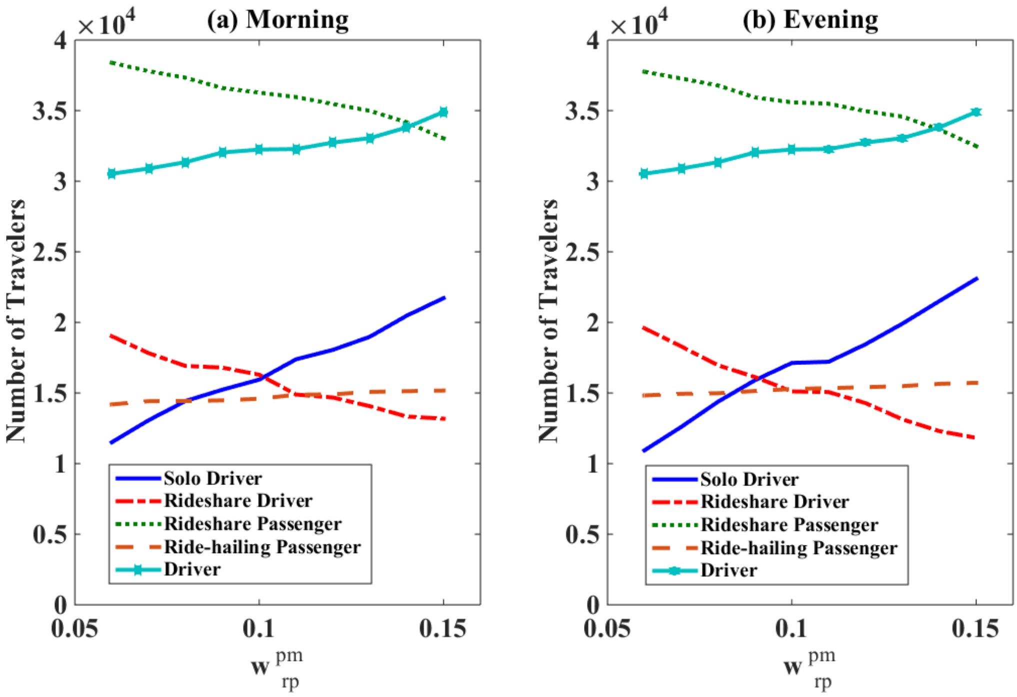

Figure 4. Results of travelers' mode choice when changing $w_{\mathrm{rp}}^{\mathrm{pm}}$.

Fig. 5 illustrates how the change of evening rideshare driver's inconvenience ($\gamma_{\mathrm{rd}}^{\mathrm{pm}}$) influences travelers' mode choice both in the morning and evening. As shown in Fig. 5(b), when the evening rideshare driver's inconvenience increases (perhaps because some drivers need to pick up their kids after work), the number of evening rideshare drivers decreases, which could cause higher rideshare payment in the evening. Consequently, there are fewer rideshare passengers in the evening. In this case, most of the rideshare drivers and passengers switch to solo drivers in the evening. Similar phenomenon can be observed in Fig. 5(a) in the morning, due to the interactions between the morning and evening commutes. With fewer rideshare drivers and passengers in the system, we can expect larger VHTs as in Fig. 3(b). Even if we only change the evening rideshare driver's inconvenience, travelers' mode choice in the morning changes more rapidly, compared with that of the evening (especially rideshare drivers). This is consistent with the faster increase of morning VHT in Fig. 3(b), which again indicates that the influence of morning (or evening) parameters to evening (or morning) model results could be significant.

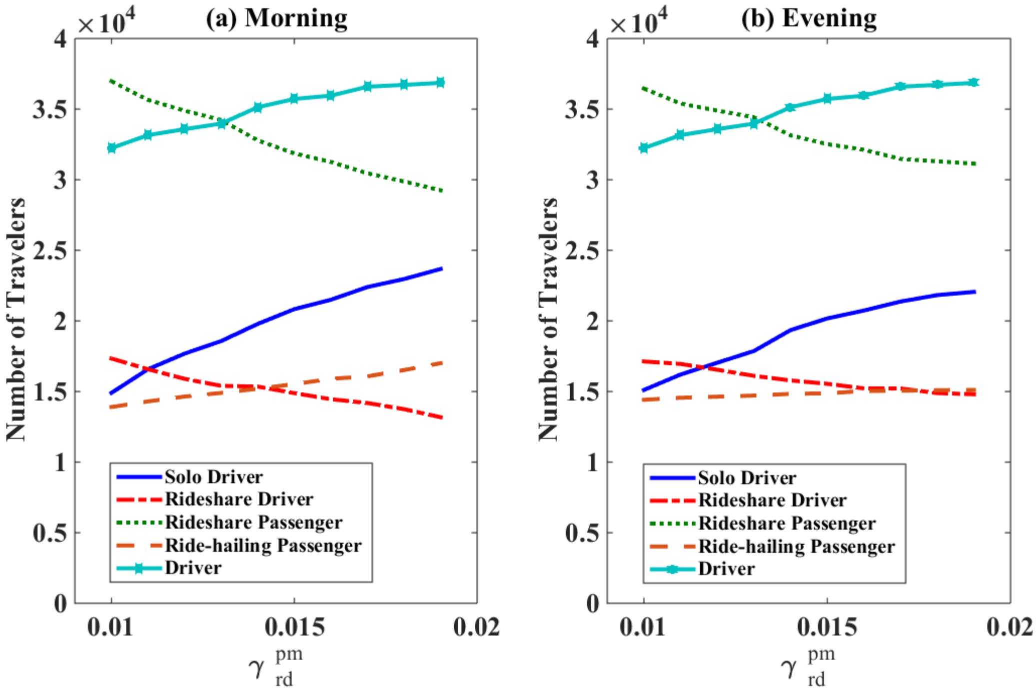

Figure 5. Results of travelers' mode choice when changing $\gamma_{\mathrm{rd}}^{\mathrm{pm}}$.

The interactions between morning and evening can also be observed from rideshare drivers' choice for detour. Fig. 6 demonstrates the proportion of rideshare drivers to choose detour when changing evening rideshare payment ($w_{\mathrm{rp}}^{\mathrm{pm}}$). As shown in the figure, when we increase $w_{\mathrm{rp}}^{\mathrm{pm}}$, more rideshare drivers are motivated to choose detour in the evening because as $w_{\mathrm{rp}}^{\mathrm{pm}}$ increases, the total detour payment increases faster than the total payment without detour in the evening. In this case, when we increase $w_{\mathrm{rp}}^{\mathrm{pm}}$ from 0.06 to 0.15, in the evening, the total detour payment in the evening increases by 2.2 dollars while the total payment without detour in the evening only increases by 0.9 dollars. As a result, 9.4% more of the evening rideshare drivers are motivated to choose detour. From the coupling effects between morning and evening, 6.9% more of the morning rideshare drivers that choose detour, even if we only change the evening parameter.

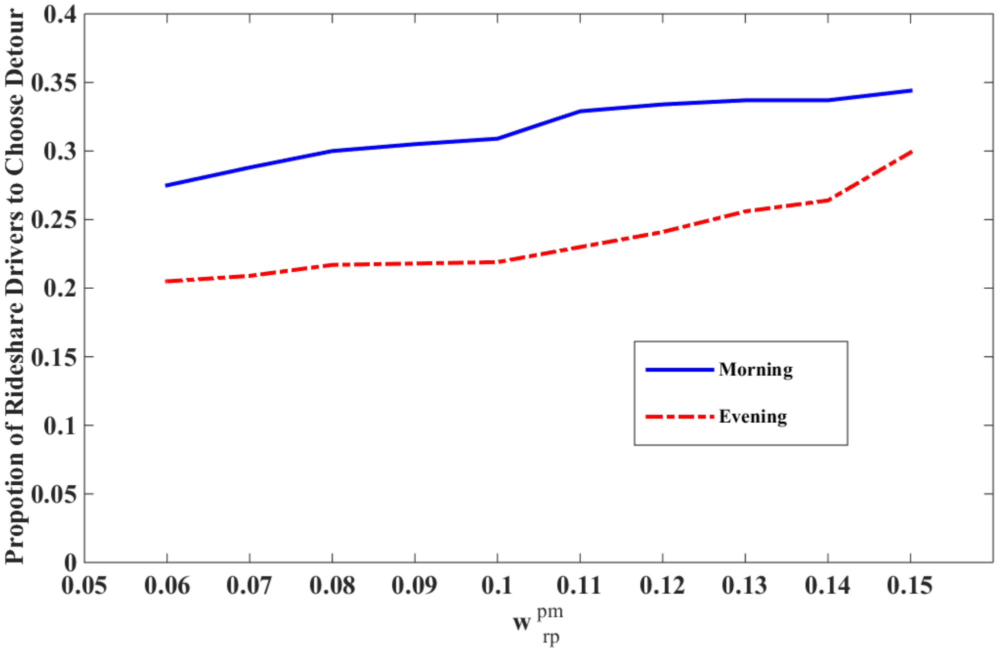

Figure 6. Results of rideshare drivers' proportion for detour when changing $w_{\mathrm{rp}}^{\mathrm{pm}}$.

We next compare the equilibrium solution from the coupled model against the decoupled approach. The decoupled solution is based on solution D2, and similar results can be found for solution D1. The main quantities for comparison include least disutility, VHT, and total number of drivers.

We use the same parameters as those in Table 4, except that $\gamma_{\mathrm{rp}}^{\mathrm{pm}}$ is 0.02 dollars. Recall $\gamma_{\mathrm{rp}}^{\mathrm{pm}}$ is 0.01 dollars. This scenario represents the case where the inconvenience cost of a rideshare passenger from work place to home during the evening commute is higher than that from home to work place in the morning commute. We assume that individuals will use the higher cost parameters to determine their mode choice in a decoupled model. Thus, in the decoupled model, since the rideshare inconvenience cost is higher in the evening and all other parameters are the same, an individual will determine whether or not to be a driver using the evening parameter settings.

The comparison between the two models is shown in Table 7. We can see that the coupled model outputs a solution with a 9.1% smaller least disutility compared with the decoupled model. The decoupled model overestimates the number of drivers by 18.2% and the VHT by 8.4% compared with the coupled model because the coupled model is capable of capturing the mode switches between morning and evening, which leads to fewer drivers and less VHT in the system. As shown in Table 5, in this case, 6.8% of the morning rideshare passengers switch to ride-hailing service in the evening because of the higher inconvenience cost of ridesharing during the evening commute, due to, for example, some individuals needing to pick up their children at their after-school activities, making the use of rideshare service during the evening less convenient. The decoupled model cannot capture this effect and most likely will predict that the traveler will drive to work, thus causing the overestimation of the number of drivers under this situation.

Table 7. Comparison between coupled model and decoupled model.

	Coupled Model	Decoupled Model
Least Disutility (Dollars)	62.45	68.70
Total VHT	23505	27783
Total Number of Drivers	32240	34948

6 Conclusions and Future Research

In this study, we include both rideshare and ride-hailing as travel modes and integrate morning and evening commute trips in a general network equilibrium modeling framework, which allows travelers to switch from one type of commute mode in the morning to another in the evening, and allows passengers from different OD pairs to share a ride together. The model is formulated as a variational inequality (VI), and reformulated as an equivalent mixed complementarity problem (MiCP). Then, we derive the conditions under which the solution will be unique, and prove that the traveler's disutility of our coupled model will not be worse than that of a decoupled modeling approach. The proposed model is evaluated on the Sioux-Falls network. The results show that the proposed coupled morning-evening traffic equilibrium model is capable of capturing the mode switches and the coupling effect between morning and evening, and the detour of rideshare drivers. Specifically, 7.0% of rideshare drivers in the morning switch to be solo drivers in the evening; the morning Vehicle Hours Traveled (VHT) increases by 12.3% even if we only change the evening rideshare payment; 30.9% of the rideshare drivers that choose to take a detour for picking up or dropping off passengers in the morning. Our numerical examples show that considering morning and evening commutes separately tends to overestimate the least disutility (travel cost), number of drivers and total VHT in the network. For example, the proposed model produces 7.8% fewer drivers and 15.4% less VHT in the system compared with a decoupled method when the rideshare price is higher in the evening commute than that of the morning commute. This is due to the coupling interaction effects between morning and evening commutes, e.g., rideshare passengers in the morning commute may switch to ride-hailing passengers in the evening commute. When treating the morning and evening commutes separately, we cannot capture these interactions.

One direction for future research could be to extend our model to formulate the trip chains of travelers throughout an entire day. In our model, we already capture some trip chains of travelers, such as the home-work-home trip chain and the detours of rideshare vehicles. However, the whole-day trip chains of travelers have not been modeled explicitly. One challenge to model the entire-day trip chains is due to the probably much more complicated formulation. For example, after returning home by driving a car, a traveler may decide to take a ride for shopping at night, which will change the total number of drivers in the system. This may eventually lead to a non-square complementarity formulation, which is mathematically difficult to analyze. Another difficulty lies in how to solve the model. With the rideshare and ride-hailing services to provide more choices for travelers, the dimension of the problem will increase exponentially with longer trip chains. As a result, it could be quite challenging to solve the model. More resource is required for developing advanced scalable algorithms. Another research direction could be to extend our static traffic equilibrium model to a dynamic one. In this case, more advanced mathematical tools such as differential variational inequalities may be needed, which could lead to a rather different model.

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8 Data Management Plan

Products of Research

The main research products will be peer-reviewed journal articles, book chapters and/or conference proceedings targeted towards the transportation science research community, plus supplemental materials such as tables, numerical data used for graphs, etc. No personal data will be used in the project, so there is no threat of identity theft.

Data Format and Content

All research products will be available online in digital form. Manuscripts will appear in a common document-viewing format, such as PDF, and supplemental materials such as tables and numerical data will be in a tabular format, such as Microsoft Excel spreadsheet, tab-delimited text, etc.

Data Access and Sharing

All participants in the project will publish the results of their work. Papers will be published in peer-reviewed scientific journals, books published in English, conference proceedings, or as peer-reviewed data reports. Beyond the data posted on USC websites, primary data and other supporting materials created or gathered in the course of work will be shared with other researchers upon reasonable request, at no more than incremental cost and within a reasonable time of the request or, if later, the filing of a patent application covering the results of such research.

All the data used in the research are included in Tables in the final report or are publicly available. For the numerial experiments, the parameters of the travel modes can be found in Table 4 and the data of the Sioux Falls can be found in the following link:
`https://github.com/bstabler/TransportationNetworks/tree/master/SiouxFalls`
Ben, S. (2020). Transportation Networks for Research

Reuse and Redistribution

USC's policy is to encourage, wherever appropriate, research data to be shared with the general public through internet access. This public access will be regulated by the university in order to protect privacy and confidentiality concerns, as well to respect any proprietary or intellectual property rights. Administrators will consult with the university's legal office to address any concerns on a case-by-case basis, if necessary. Terms of use will include requirements of attribution along with disclaimers of liability in connection with any use or distribution of the research data, which may be conditioned under some circumstances.

Appendix 1. The equivalent mixed complementarity formulation of the arc-based VI model.

$$0 \le x^{\mathrm{am}}_{a;\,k} \perp \mathrm{tc}^{\mathrm{am}}_a(\boldsymbol{x}^{\mathrm{am}}_k) - \mu^{k;\mathrm{am}}_i + \mu^{k;\mathrm{am}}_j - \zeta_{a;\,k} \ge 0, \quad \forall a=(i,j)\in\mathcal{A}_{\mathrm{sd}}, \forall k\in\mathcal{K}$$

$$0 \le x^{\mathrm{am}}_{a;\,k} \perp \mathrm{tc}^{\mathrm{am}}_a(\boldsymbol{x}^{\mathrm{am}}_k) + \eta^{+;\mathrm{am}}_{\mathcal{T}_0(a)} - M\eta^{-;\mathrm{am}}_{\mathcal{T}_0(a)} - \mu^{k;\mathrm{am}}_i + \mu^{k;\mathrm{am}}_j - \zeta_{a;\,k} \ge 0, \quad \forall a=(i,j)\in\mathcal{A}_{\mathrm{rd}}, \forall k\in\mathcal{K}$$

$$0 \le x^{\mathrm{am}}_{a;\,k} \perp \mathrm{tc}^{\mathrm{am}}_a(\boldsymbol{x}^{\mathrm{am}}_k) - \eta^{+;\mathrm{am}}_{\mathcal{T}_0(a)} + \eta^{-;\mathrm{am}}_{\mathcal{T}_0(a)} - \mu^{k;\mathrm{am}}_i + \mu^{k;\mathrm{am}}_j \ge 0, \quad \forall a=(i,j)\in\mathcal{A}_{\mathrm{rp}}, \forall k\in\mathcal{K}$$

$$0 \le x^{\mathrm{am}}_{a;\,k} \perp \mathrm{tc}^{\mathrm{am}}_a(\boldsymbol{x}^{\mathrm{am}}_k) - \mu^{k;\mathrm{am}}_i + \mu^{k;\mathrm{am}}_j \ge 0, \quad \forall a=(i,j)\in\mathcal{A}_{\mathrm{hp}}, \forall k\in\mathcal{K}$$

$$0 \le x^{\mathrm{pm}}_{a;\,\bar{k}} \perp \mathrm{tc}^{\mathrm{pm}}_a(\boldsymbol{x}^{\mathrm{pm}}_{\bar{k}}) - \mu^{\bar{k};\mathrm{pm}}_i + \mu^{\bar{k};\mathrm{pm}}_j + \zeta_{a;\,\bar{k}} \ge 0, \quad \forall a=(i,j)\in\mathcal{A}_{\mathrm{sd}}, \ \text{associated } \bar{k}$$

$$0 \le x^{\mathrm{pm}}_{a;\,\bar{k}} \perp \mathrm{tc}^{\mathrm{pm}}_a(\boldsymbol{x}^{\mathrm{pm}}_{\bar{k}}) + \eta^{+;\mathrm{pm}}_{\mathcal{T}_0(a)} - M\eta^{-;\mathrm{pm}}_{\mathcal{T}_0(a)} - \mu^{\bar{k};\mathrm{pm}}_i + \mu^{\bar{k};\mathrm{pm}}_j + \zeta_{a;\,\bar{k}} \ge 0, \quad \forall a=(i,j)\in\mathcal{A}_{\mathrm{rd}}, \ \text{associated } \bar{k}$$

$$0 \le x^{\mathrm{pm}}_{a;\,\bar{k}} \perp \mathrm{tc}^{\mathrm{pm}}_a(\boldsymbol{x}^{\mathrm{pm}}_{\bar{k}}) - \eta^{+;\mathrm{pm}}_{\mathcal{T}_0(a)} + \eta^{-;\mathrm{pm}}_{\mathcal{T}_0(a)} - \mu^{\bar{k};\mathrm{pm}}_i + \mu^{\bar{k};\mathrm{pm}}_j \ge 0, \quad \forall a=(i,j)\in\mathcal{A}_{\mathrm{rp}}, \ \text{associated } \bar{k}$$

$$0 \le x^{\mathrm{pm}}_{a;\,\bar{k}} \perp \mathrm{tc}^{\mathrm{pm}}_a(\boldsymbol{x}^{\mathrm{pm}}_{\bar{k}}) - \mu^{\bar{k};\mathrm{pm}}_i + \mu^{\bar{k};\mathrm{pm}}_j \ge 0, \quad \forall a=(i,j)\in\mathcal{A}_{\mathrm{hp}}, \ \text{associated } \bar{k}$$

$$\left.\begin{array}{lll}
0 \le \eta^{+;\mathrm{am}}_a & \perp & \sum_{k\in\mathcal{K}} x^{\mathrm{am}}_{a_{\mathrm{rp}};\,k} - \sum_{k\in\mathcal{K}} x^{\mathrm{am}}_{a_{\mathrm{rd}};\,k} \ge 0 \\
0 \le \eta^{-;\mathrm{am}}_a & \perp & M\sum_{k\in\mathcal{K}} x^{\mathrm{am}}_{a_{\mathrm{rd}};\,k} - \sum_{k\in\mathcal{K}} x^{\mathrm{am}}_{a_{\mathrm{rp}};\,k} \ge 0 \\
0 \le \eta^{+;\mathrm{pm}}_a & \perp & \sum_{\bar{k}\in\bar{\mathcal{K}}} x^{\mathrm{pm}}_{a_{\mathrm{rp}};\,\bar{k}} - \sum_{\bar{k}\in\bar{\mathcal{K}}} x^{\mathrm{pm}}_{a_{\mathrm{rd}};\,\bar{k}} \ge 0 \\
0 \le \eta^{-;\mathrm{pm}}_a & \perp & M\sum_{\bar{k}\in\bar{\mathcal{K}}} x^{\mathrm{pm}}_{a_{\mathrm{rd}};\,\bar{k}} - \sum_{\bar{k}\in\bar{\mathcal{K}}} x^{\mathrm{pm}}_{a_{\mathrm{rp}};\,\bar{k}} \ge 0
\end{array}\right\} \begin{array}{l}\forall\, a \in \mathcal{A}_0,\ a_{\mathrm{rd}} = \mathcal{T}_{\mathrm{rd}}(a), \\ \text{and } a_{\mathrm{rp}} = \mathcal{T}_{\mathrm{rp}}(a)\end{array}$$

$$\mu^{k;\mathrm{am}}_i, \mu^{\bar{k};\mathrm{pm}}_i \ \text{free} \ \perp \left\{\begin{array}{l}
\sum_{a\in\mathrm{IN}(d_k)} x^{\mathrm{am}}_{a;\,k} = \sum_{a\in\mathrm{OUT}(o_k)} x^{\mathrm{am}}_{a;\,k} = \sum_{a\in\mathrm{IN}(d_{\bar{k}})} x^{\mathrm{pm}}_{a;\,\bar{k}} = \sum_{a\in\mathrm{OUT}(o_{\bar{k}})} x^{\mathrm{pm}}_{a;\,\bar{k}} = D_k \\
\forall k\in\mathcal{K} \text{ and associated } \bar{k} \\
\sum_{a\in\mathrm{IN}(i)} x^{\mathrm{am}}_{a;\,k} - \sum_{a\in\mathrm{OUT}(i)} x^{\mathrm{am}}_{a;\,k} = 0 \quad \forall i\in\mathcal{N}\setminus\{o_k, d_k\}, \forall k\in\mathcal{K} \\
\sum_{a\in\mathrm{IN}(i)} x^{\mathrm{pm}}_{a;\,\bar{k}} - \sum_{a\in\mathrm{OUT}(i)} x^{\mathrm{pm}}_{a;\,\bar{k}} = 0 \quad \forall i\in\mathcal{N}\setminus\{o_{\bar{k}}, d_{\bar{k}}\}, \ \text{associated } \bar{k}
\end{array}\right.$$

$$\zeta_{a;\,k} \ \text{free} \ \perp \sum_{a_{\mathrm{sd}}, a_{\mathrm{rd}}\in\mathrm{OUT}(o_k)} (x^{\mathrm{am}}_{a_{\mathrm{sd}};\,k} + x^{\mathrm{am}}_{a_{\mathrm{rd}};\,k}) = \sum_{a_{\mathrm{sd}}, a_{\mathrm{rd}}\in\mathrm{IN}(d_{\bar{k}})} (x^{\mathrm{pm}}_{a_{\mathrm{sd}};\,\bar{k}} + x^{\mathrm{pm}}_{a_{\mathrm{rd}};\,\bar{k}}) \quad \forall k\in\mathcal{K} \text{ and associated } \bar{k}$$